231 Outline Solutions Tutorial Sheet 13, 14 and 15.¹²

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Problem Sheet 13

1. Express the following functions as Fourier integrals:

(a)

(b)

 $f(x) = \begin{cases} \cos x & |x| < \frac{\pi}{2} \\ 0 & |x| > \frac{\pi}{2} \end{cases}$ $f(x) = \frac{\sin x}{r}$

Solution:(a) Writing f as a Fourier integral $f(x) = \int_{-\infty}^{\infty} dk \ e^{ikx} \ \tilde{f}(k)$. We require the Fourier transform:

$$\begin{split} \tilde{f}(k) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \; e^{-ikx} \; f(x) = \frac{1}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dk \; e^{-ikx} \; \frac{e^{ix} + e^{-ix}}{2} \\ &= \frac{1}{4\pi} \left(\frac{e^{i(1-k)x}}{i(1-k)} + \frac{e^{i(-1-k)x}}{i(-1-k)} \right) \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\ &= \frac{1}{4\pi} \left[\frac{ie^{-ik\pi/2} + ie^{ik\pi/2}}{i(1-k)} + \frac{-ie^{-ik\pi/2} - ie^{ik\pi/2}}{i(-1-k)} \right] \\ &= \frac{1}{4\pi} \; 2\cos\left(\frac{k\pi}{2}\right) \; \left(\frac{1}{1-k} + \frac{1}{1+k}\right) = \frac{1}{\pi}\cos\left(\frac{k\pi}{2}\right) \; \frac{1}{1-k^2}. \end{split}$$

Therefore

$$f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} dk \, \cos\left(\frac{k\pi}{2}\right) \, \frac{e^{ikx}}{1 - k^2}.$$

Remark: $\tilde{f}(k)$ is well behaved at $k = \pm 1$. (b)

$$\frac{\sin x}{x} = \frac{1}{2} \int_{-1}^{1} dk \ e^{ikx}$$

Remark: In the lectures it was shown that the Fourier transform of a square pulse is proportional to $\sin k/k$ and so it follows that the Fourier transform of the $\sin x/x$ is proportional to the pulse and, for example, integrating quickly gives the constant of proportionality.

- 2. Prove the following properties of the Fourier transform
 - (a) The Fourier transform of an even function is even.
 - (b) The Fourier transform of a real odd function is purely
 - (c) $\tilde{f}'(k) = ik\tilde{f}(k)$.
 - (d) Acting with the Fourier transform four times reproduces the original function apart from an overall constant.

Solution:(a) Assume that f is even, i.e. f(-x) = f(x), then

$$\tilde{f}(-k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \ e^{ikx} f(x).$$

make the change of variables y = -x:

$$\tilde{f}(-k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dy \ e^{-iky} f(-y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dy \ e^{-iky} f(y) = \tilde{f}(k)$$

(b) Assume that f is real and odd, i.e. f(-x) = -f(x) and $\bar{f}(x) = f(x)$

$$\overline{\tilde{f}(k)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \ e^{ikx} \ f(x).$$

Make the change of variables y = -x

$$\overline{\tilde{f}(k)} = \frac{1}{2\pi} \int_{\infty}^{-\infty} (-dy) \ e^{-iky} f(-y) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} (dy) \ e^{-iky} f(y) = -\tilde{f}(k).$$

(c) here an integration by parts is required

$$\tilde{f}'(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \ e^{-ikx} f'(x) = e^{-ikx} f(x) \big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} dx (-ik) e^{-ikx} f(x) = ik \tilde{f}(k).$$

assuming that the boundary terms vanish.

(d) The Fourier integral representation of a function f, i.e.

$$f(x) = \int_{-\infty}^{\infty} dk \ e^{ikx} \ \tilde{f}(k),$$

can be written as

 $f(-x) = 2\pi \tilde{\tilde{f}}(x),$

or $\tilde{f}(x) = f(-x)/(2\pi)$. Acting with the Fourier transform twice reproduces the original function up to the $x \to -x$ reflection multiplied by $1/(2\pi)$. Acting with the Fourier transform four times reproduces the original function multiplied by $(2\pi)^{-2}$.

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3. Compute

(c)

(d)

(e)

(a)
$$\int_{-\infty}^{\infty} dx \ x^2 \ \delta(x)$$
 (b)

$$\int_{-\infty}^{\infty} dx \ \delta(x^2 + x)$$

$$\int_0^2 dx \ e^x \ \delta'(x-1)$$

(-3)

$$\int_0^\infty dx \ e^{-ax} \delta(\cos x)$$
$$\int_0^\infty dx \ \delta(e^{ax} \cos x).$$

(f)

$$\frac{d}{dx}e^{a\theta(x)}.$$

where a is a constant.

Solution: (a) $\int_{-\infty}^{\infty} dx \ x^2 \delta(x-3) = 3^2 = 9.$ (b) Use

$$\delta(h(x)) = \sum_{i} \frac{\delta(x - x_i)}{|h'(x_i)|},$$

where the x_i are roots of h. Here $h(x) = x^2 + x = x(x+1)$ with roots $x_1 = 0$ and $x_2 = -1$. h'(x) = 2x + 1 and so h'(0) = 1, h'(-1) = -1. This gives $\delta(x^2 + x) = \delta(x) + \delta(x+1)$

$$\int_{-\infty}^{\infty} dx \ \delta(x^2 + x) = 2$$

(c) Integrate by parts:

$$\int_0^2 dx \ e^x \ \delta'(x-1) = \ e^x \delta(x-1)|_0^2 - \int_0^2 dx \ e^x \ \delta(x-1) = -e.$$

(d) $h(x) = \cos x$ has zeros at $x = \frac{1}{2}\pi$, $\frac{3}{2}\pi$, $\frac{5}{2}\pi$ etc. and the derivative of $\cos x$ is equal to 1 or -1 at these points. Therefore

$$\int_0^\infty dx \ e^{-ax} \delta(\cos x) = \sum_{n=0}^\infty \ e^{-a(\frac{1}{2}\pi + n\pi)} = e^{-\frac{1}{2}a\pi} \sum_{n=0}^\infty \ e^{-an\pi} = \frac{e^{-\frac{1}{2}a\pi}}{(1 - e^{-a\pi})},$$
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the last step used the standard geometric series formula. The result may be rewritten in terms of the hyperbolic sine.

$$\int_0^\infty dx \ e^{-ax} \delta(\cos x) = \frac{1}{2\sinh\frac{1}{2}a\pi}.$$

(e) $h(x) = e^{ax} \cos x$, $h'(x) = ae^{ax} \cos x - e^{ax} \sin x$. The zeros of h are the same as in the previous problem. At a zero $|h'(x)| = e^{ax}$. This implies that the integral leads to the same geometric sum as in part (d). (f) First you need to reexpress everything so that it is linear in $\theta(x)$, we can't differenciate powers of $\theta(x)$. So

$$\exp a\theta = \sum_{n=0}^{\infty} \theta(x)^n a^n n!$$
$$= 1 + \sum_{n=1}^{\infty} \theta(x)^n a^n n!$$

then, using $\theta^n = \theta$, easy to check from the definition of θ , we get

$$\exp a\theta = 1 + \theta \sum_{n=1}^{\infty} \frac{a^n}{n!}$$
$$= 1 + \theta \left(\sum_{n=0}^{\infty} \frac{\theta(x)^n a^n}{n!} - 1 \right)$$
$$= 1 + \theta \left(e^a - 1 \right)$$

and hence

$$\frac{d}{dx}e^{a\theta(x)} = \frac{d}{dx}\left[1 + \theta\left(e^a - 1\right)\right] = \delta(x)\left(e^a - 1\right)$$

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Problem Sheet 14

1. In the lectures (quite a while ago) it was shown that the scalar field

$$\phi(\mathbf{r}) = \frac{1}{r},$$

where $r=\sqrt{x^2+y^2+z^2}$ is harmonic except at the origin. In fact it can be shown that

$$\nabla^2 \phi(\mathbf{r}) = -4\pi \delta^3(\mathbf{r}). \tag{A}$$

Formally apply Gauss' theorem to the vector field $\mathbf{F} = \nabla \phi$ to show that

$$\int_{r < a} dV \,\nabla^2 \phi = -4\pi.$$

This is clearly consistent with (A). Another treatment would replace the singular scalar field ϕ with a sequence of smooth scalar fields, e.g.

$$\phi_n(\mathbf{r}) = \frac{n}{\sqrt{n^2 r^2 + 1}}.$$

Prove that

$$\int_{R^3} dV \,\nabla^2 \phi_n(\mathbf{r}) = -4\pi.$$

Solution: $\mathbf{F} = \nabla \phi$ so that $\nabla^2 \phi = \text{div } \mathbf{F}$. Applying Gauss' theorem

$$\int_{r < a} dV \nabla^2 \phi = \int_{r < a} dV \operatorname{div} \mathbf{F} = \int_{r = a} \mathbf{F} \cdot \mathbf{dA}.$$

 $\mathbf{F} = -\mathbf{r}/r^3$ and $\mathbf{F} \cdot \mathbf{n} = -1/a^2$ and the surface are is $4\pi a^2$ giving

$$\int_{r< a} dV \,\nabla^2 \phi = -4\pi.$$

 $\partial_x \phi_n = -\frac{1}{2}n(n^2r^2+1)^{-3/2}2xn^2$, and similarly for $\partial_y \phi_n$ and $\partial_z \phi_n$. Therefore

$$\nabla \phi_n = -\frac{n^3 \mathbf{r}}{(n^2 r^2 + 1)^{3/2}}.$$
$$\int_{r < a} dV \ \nabla^2 \phi_n = \int_{r=a} \nabla \phi_n \cdot \mathbf{dA} = -\frac{n^3 \ 4\pi a^3}{(n^2 a^2 + 1)^{3/2}} \to -4\pi$$
as $a \to \infty$.

Problem Sheet 15

1. Inside an integral, what is

$$\frac{d}{dx}\frac{1}{1+\epsilon\theta(x)}\tag{1}$$

for $\theta(x)$ the usual Heaviside function and $|\epsilon| < 1$.

Solution: There are two ways to do this, we can either expand the fraction as a power series or we can try and evaluate it inside an integral. First the first way, using the usual expansion of 1/(1+x) for x < 1

$$\frac{1}{1+\epsilon\theta(x)} = \sum_{n=0}^{\infty} \left[-\epsilon\theta(x)\right]^n \tag{2}$$

Now, we just use the fact that $\theta(x)^n = \theta(x)$ for n a positive integer; note that we have to be careful with the first term in the series which doesn't contain a $\theta(x)$ factor since n is zero. Hence

$$\frac{1}{1+\epsilon\theta(x)} = 1+\theta(x)\sum_{n=1}^{\infty} (-\epsilon)^n = 1+\theta(x)\sum_{n=0}^{\infty} (-\epsilon)^n - \theta(x) = 1-\theta(x) + \frac{1}{1+\epsilon}\theta(x)$$
(3)

where we have added and taken away the missing term in the sum. Hence,

$$\frac{d}{dx}\frac{1}{1+\epsilon\theta(x)} = \frac{d}{dx}[1-\theta(x) + \frac{1}{1+\epsilon}\theta(x)] = \frac{1}{1+\epsilon}\delta(x) - \delta(x)$$
(4)

The other way is to do the calculation inside an integral: with a < 0 and b > 0

$$I = \int_{a}^{b} dx f(x) \frac{d}{dx} \frac{1}{1 + \epsilon \theta(x)}$$
(5)

Now, integrating by parts

$$I = f'(x)\frac{1}{1+\epsilon\theta(x)}\bigg]_a^b - \int_a^b dx f'(x)\frac{1}{1+\epsilon\theta(x)}$$
(6)

Next, we split the integral into two and use the fact $\theta(x)$ is zero for negative x and one for positive x. So,

$$I = \frac{1}{1+\epsilon}f(b) - f(a) - \int_{a}^{0} dx f'(x) - \frac{1}{1+\epsilon} \int_{0}^{b} dx f'(x)$$
(7)

and, using the Fundamental Theorem of Calculus

$$I = \frac{1}{1+\epsilon}f(b) - f(a) - f(0) + f(a) - \frac{1}{1+\epsilon}f(b) + \frac{1}{1+\epsilon}f(0) = \left(\frac{1}{1+\epsilon} - 1\right)f(0) \quad (8)$$

which implies

$$\frac{d}{dx}\frac{1}{1+\epsilon\theta(x)} = \frac{1}{1+\epsilon}\delta(x) - \delta(x) \tag{9}$$

as before.

2. Compute

(a) $\int_{-\infty}^{\infty} dx \ e^x \ \delta(x+1)$ (b) $\int_{-3}^{1} dx \ \delta(x^2 - 3x + 2)$ (c) $\int_{-\infty}^{\infty} dx \ \cos x \ \delta'(x)$ (d) $\int_{0}^{1} dx \ \delta\left(\sin \frac{1}{x}\right).$

Solution:

(a) $\int_{-\infty}^{\infty} dx \ e^x \ \delta(x+1) = e^{-1}$. (b) Use

$$\delta(h(x)) = \sum_{i} \frac{\delta(x - x_i)}{|h'(x_i)|},$$

where the x_i s are roots of h. In this case $h(x) = x^2 - 3x + 2 = (x - 2)(x - 1)$ with roots $x_1 = 2$ and $x_2 = 1$. This is a problem since x = 1 is one of the limits of integration, in fact

$$\int_{-\infty}^{0} dx \delta(x) \tag{10}$$

isn't defined, and so the answer here is that the integral isn't defined. Say instead we had been asked

$$\int_{-3}^{3} dx \,\,\delta(x^2 - 3x + 2) \tag{11}$$

then both roots are in the integral and we would use h'(x) = 2x - 3 so that h'(1) = -1, giving |h(1)| = 1 and h'(2) = 1 which gives

$$\delta(x^2 - 3x + 2) = \delta(x + 1) + \delta(x_2)$$

and

$$\int_{-3}^{3} dx \,\,\delta(x^2 - 3x + 2) = 2.$$

(c)

$$\int_{-\infty}^{\infty} dx \, \cos x \delta'(x) = -\int_{-\infty}^{\infty} dx \, (-\sin x) \, \delta(x) = 0$$

Integrating by parts and using $\sin 0 = 0$.

(d) Use formula for $\delta(h(x))$, here $h(x) = \sin(1/x)$ which is zero for $1/x = n\pi$ $(n \in Z)$. $h'(x) = -x^{-2}\cos(1/x)$ and since $|\cos n\pi| = 1$

$$\delta(h(x)) = \sum_{n \neq 0} \frac{\delta\left(x - \frac{1}{n\pi}\right)}{\pi^2 n^2}.$$

Now $1/(n\pi) \in (0,1)$ for all positive n which gives

$$\int_{0}^{1} dx \, \delta\left(\sin\frac{1}{x}\right) = \frac{1}{\pi^{2}} \sum_{n>0} \frac{1}{n^{2}}.$$

The sum on the RHS is $\zeta(2) = \pi^2/6$ (see Q3 Sheet 11) and so

$$\int_0^1 dx \ \delta\left(\sin\frac{1}{x}\right) = \frac{1}{6}.$$

3. Obtain a general solution to

(a)
$$y' - 3y = e^{-x}$$

(b) $y' + y \cot x = \cos x$
(c) $(x+1)y' + y = (x+1)^2$

Solution:

or

(a) Rewrite as

 $e^{-3x}y' - 3ye^{-3x} = e^{-4x}$

 $\left(e^{-3x}y\right)' = e^{-4x}$

 $(\sin xy)' = (\sin^2 x)'$

and then integrate.

(b) the quuickest thing to do is multiply across by the sine

$$\sin xy' + \cos xy = \sin x \cos x \tag{12}$$

and rewritting

hence

 $\sin xy = \sin^2 x + C \tag{14}$

(13)

or

$$y = \sin x + C \operatorname{cosec} x \tag{15}$$

(c)
$$(x+1)y' + y = (x+1)^2$$
 can again be rewritten

$$[(x+1)y]' = x^2 + 2x + 1 \tag{16}$$

 \mathbf{SO}

or

$$(x+1)y = \frac{1}{3}x^3 + x^2 + x + C \tag{17}$$

$$3y = \frac{x^3 + 3x^2 + 3x + 1}{x+1} + \frac{C}{x+1} = (x+1)^2 + \frac{C}{x+1}$$
(18)

with a redefinition of C to get the nice devision at the end, another way to do this would have been to change variables to z = x + 1 at the start.

4. Obtain the general solutions of the following ODEs:

(a) y'' + 5y' + 6y = 0(b) y'' - 2y' + y = 0

Solution:

(a) y'' + 5y' + 6y = 0 so substitute $e^{\lambda x}$ to get the auxiliary equation

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$$\lambda^2 + 5\lambda + 6 = 0 \tag{19}$$

so $\lambda = -2$ and $\lambda = -3$ giving solution

$$y = C_1 e^{-2x} + C_2 e^{-3x} \tag{20}$$

(b) y'' - 2y' + y = 0 gives auxiliary equation

$$\lambda^2 - 2\lambda + 1 = 0 \tag{21}$$

which has $\lambda = 1$ as a repeated root, so

$$y = C_1 e^x + C_2 x e^x \tag{22}$$