231 Outline Solutions Tutorial Sheet 1, 2 and 3.[1][2]

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Problem Sheet 1

1. Rewrite the integral

$$I = \int_0^1 dx \int_1^{e^x} dy\ \phi(x, y) \tag{1}$$

as a double integral with the opposite order of integration.

*Solution:*The range of y values: $1 \le y \le e$. For a fixed y, x has the range $\log y \le x \le 1$. Hence

$$I = \int_1^e dy \int_{\log y}^1 dx\ \phi(x, y). \tag{2}$$

2. Evaluate

$$I = \int_D dxdyxe^{xy} \tag{3}$$

where D is given by $0 < x < 1$ and $2 < y < 4$.

*Solution:*So rewriting as an iterated integral

$$I = \int_D dxdyxe^{xy} = \int_0^1 dx \int_2^4 dyxe^{xy} \tag{4}$$

and integrating from the middle

$$\int_0^1 dx \int_2^4 dyxe^{xy} = \int_0^1 dx \left(x\frac{1}{x}e^{xy} \right)_2^4 = \int_0^1 dx \left(e^{4x} - e^{2x} \right) = \frac{1}{4}e^4 - \frac{1}{2}e^2 + \frac{1}{4} \tag{5}$$

Here we cunningly made the integration easier by doing the y integration first, in fact is shouldn't make any difference to the answer if the integration is done in the other order, it is definately harder thought:

$$I = \int_D dxdyxe^{xy} = \int_2^4 dy \int_0^1 dxxe^{xy} = \int_2^4 dy \left(\frac{1}{y^2} - \frac{1}{y^2}e^y + \frac{1}{y}e^y \right) \tag{6}$$

where we did the x integral using integration by parts, Now integrating by parts

$$\int_2^4 dy\frac{1}{y}e^y = \frac{1}{y}e^y]_2^4 + \int_2^4 dy\frac{1}{y^2}e^y \tag{7}$$

[1]Conor Houghton, `houghton@maths.tcd.ie`, see also `http://www.maths.tcd.ie/~houghton/231`

[2]Including material from Chris Ford, to whom many thanks.

so, substituting this in and cancelling

$$\int_2^4 dy\left(\frac{1}{y^2}-\frac{1}{y^2}e^y+\frac{1}{y}e^y\right)=\frac{1}{y}e^y]_2^4+\int_2^4 dy\left(\frac{1}{y^2}\right)=\frac{1}{4}e^4-\frac{1}{2}e^2+\frac{1}{4} \tag{8}$$

as before.

3. Evaluate

$$I=\int_D dxdy(x+y) \tag{9}$$

where D is given by $0<y<1$ and $2y<x<2$.

Solution: So, write as an iterated integral

$$I=\int_D dxdy(x+y)=\int_0^1 dy\int_{2y}^2 dx(x+y) \tag{10}$$

and integrate from the inside out

$$\begin{aligned}\int_0^1 dy\int_{2y}^2 dx(x+y) &= \int_0^1 dy\left(\frac{1}{2}x^2+xy\right)_{2y}^2=\int_0^1 dy\left(2+2y-4y^2\right)\\ &= 3-\frac{4}{3}=\frac{5}{3}\end{aligned} \tag{11}$$

4. Change the order of integration of

$$I=\int_0^{1/2} dy\int_{-\sqrt{1-4y^2}}^{\sqrt{1-4y^2}} dxy \tag{12}$$

and evaluate.

Solution: So

$$x=\pm\sqrt{1-4y^2} \tag{13}$$

implies

$$y=\pm\frac{1}{2}\sqrt{1-x^2} \tag{14}$$

and it is easy to see from drawing a picture that

$$I=\int_0^{1/2} dy\int_{-\sqrt{1-4y^2}}^{\sqrt{1-4y^2}} dxy=\int_{-1}^1 dx\int_0^{\frac{1}{2}\sqrt{1-x^2}} dyy \tag{15}$$

Now, integrating we get

$$\int_{-1}^1 dx\int_0^{\frac{1}{2}\sqrt{1-x^2}} dyy=\frac{1}{8}\int_{-1}^1 dx(1-x^2)=\frac{1}{8}\left(2-\frac{2}{3}\right)=\frac{1}{6} \tag{16}$$

Problem Sheet 2

1. Consider the integral
$$I = \int_D dV\ \phi \tag{17}$$
where D is the interior of the ellipsoid defined by
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1. \tag{18}$$
Write down I as an iterated triple integral.

*Solution:*Upper surface of ellipsoid ia
$$z = +c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}. \tag{19}$$
whereas the lower surface is
$$z = -c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}} \tag{20}$$
The surfaces join at $z = 0$ where $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, this provides range of x and y integrations: $y = -b\sqrt{1 - \frac{x^2}{a^2}}$ to $y = +b\sqrt{1 - \frac{x^2}{a^2}}$ and $x = -a$ to $x = a$:
$$I = \int_{-a}^{a} dx \int_{-b\sqrt{1-\frac{x^2}{a^2}}}^{+b\sqrt{1-\frac{x^2}{a^2}}} dy \int_{-c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}^{+c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz\ \phi(x, y, z). \tag{21}$$

2. The Gaussian integral formula
$$\int_{-\infty}^{\infty} dx\ e^{-x^2} = \sqrt{\pi} \tag{22}$$
can be derived easily with the help of polar coordinates. The trick is to note that the *square* of the integral can be recast as a double integral over R^2:
$$\left(\int_{-\infty}^{\infty} dx\ e^{-x^2}\right)^2 = \int_{R^2} dA\ e^{-x^2-y^2}. \tag{23}$$
By changing to polar coordinates evaluate this integral.

*Solution:*After changing to polars and making sure to include the Jacobian $J = r$
$$\int_{R^2} dA\ e^{-x^2-y^2} = \int_0^{2\pi} d\theta \int_0^{\infty} dr\ re^{-r^2} \tag{24}$$
and then do this integral by substituting $u = r^2$ so $du = 2rdr$ to give
$$I^2 = \pi \int_0^{\infty} due^{-u} = \pi \tag{25}$$
as required.

3. Compute the Jacobian of the transformation from cartesian to parabolic cylinder coordinates

$$x = \frac{1}{2}(u^2 - v^2), \qquad y = uv. \tag{26}$$

Solution: Well

$$\begin{aligned} J &= \left|\left| \begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{matrix} \right|\right| \\ &= \left|\left| \begin{matrix} u & -v \\ v & u \end{matrix} \right|\right| \\ &= u^2 + v^2. \end{aligned} \tag{27}$$

4. Determine the volume of the region enclosed by the cylinder $x^2 + y^2 = 4$ and the planes $y + z = 4$ and $z = 0$. *Suggestion: Use Cartesian coordinates.*

   Solution: Range of integration: $z = 0$ to $z = 4 - y$, $y = -\sqrt{4 - x^2}$ to $y = +\sqrt{4 - x^2}$ and $x = -2$ to $x = 2$. Thus the volume is

$$V = \int_{-2}^{2} dx \int_{-\sqrt{4-x^2}}^{+\sqrt{4-x^2}} dy \int_{0}^{4-y} dz \; 1 = \int_{-2}^{2} dx \int_{-\sqrt{4-x^2}}^{+\sqrt{4-x^2}} dy \; (4 - y), \tag{28}$$

   the z integral being trivial. The y integral is also straightforward:

$$V = \int_{-2}^{2} dx \; 8\sqrt{4 - x^2} = 8 \cdot 2\pi = 16\pi. \tag{29}$$

   The final integral can be evaluated by elementary means: either make the standard substitution ($x = 2\sin\theta$) or simply note that the integral represents the area of a semi-circle of radius 2.

Problem Sheet 3

1. Rewrite the integral

$$I = \int_{0}^{1} dy \int_{\tan^{-1} y}^{\frac{\pi}{4}} dx \; \phi(x, y), \tag{30}$$

   as an iterated double integral with the opposite order of integration. Compute the area of the region of integration.

   Solution: Here $x = \frac{1}{4}\pi$ is the right boundary and $x = \tan^{-1} y$ is the left boundary. A quick sketch shows that the left boundary is also the upper boundary which can be written $y = \tan x$. The lower boundary is $y = 0$ and $0 \leq x \leq \frac{1}{4}\pi$.Thus

$$I = \int_{0}^{\frac{1}{4}\pi} dx \int_{0}^{\tan x} dy \; \phi(x, y). \tag{31}$$

Area obtained by setting $\phi(x,y)=1$:

$$
\begin{aligned}
A &= \int_0^{\frac{1}{4}\pi} dx \int_0^{\tan x} dy \\
&= \int_0^{\frac{1}{4}\pi} dx \ \tan x = -\log(\cos x)|_0^{\frac{1}{4}\pi} \\
&= -\left(\log\frac{1}{\sqrt{2}} - \log 1\right) \\
&= \frac{\log 2}{2}. \qquad (32)
\end{aligned}
$$

2. Compute the element of area for elliptic cylinder coordinates which are defined as

$$
\begin{aligned}
x &= a\cosh u \cos v \qquad (33)\\
y &= a\sinh u \sin v. \qquad (34)
\end{aligned}
$$

Solution: $\delta A = J\delta u \delta v$ with

$$
\begin{aligned}
J &= \left\|\begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{matrix}\right\| = \left\|\begin{matrix} a\sinh u\cos v & -a\cosh u \sin v \\ a\cosh u \sin v & a\sinh u\cos v\end{matrix}\right\| \\
&= a^2(\sinh^2 u\cos^2 v + \cosh^2 u \sin^2 v) \qquad (35)
\end{aligned}
$$

This can be simplified a bit:

$$
\begin{aligned}
J &= a^2(\sinh^2 u\cos^2 v + \cosh^2 u\sin^2 v) = a^2[\sinh^2 u(1-\sin^2 v) + \cosh^2 u \sin^2 v] \\
&= a^2[\sinh^2 + \sin^2 v(\cosh^2 - \sinh^2 u)] \qquad (36)
\end{aligned}
$$

Using $\cosh^2 u - \sinh^2 u = 1$ gives $J = a^2(\sinh^2 u + \sin^2 v)$.

3. Compute the area and centroid of the plane region enclosed by the cardioid $r(\theta) = 1+\cos\theta$ (r and θ are polar coordinates).

   Solution: Use polar coördinates to evaluate area integral; θ ranges from 0 to 2π and r ranges from 0 to $1+\cos\theta$ and the Jacobian is $J=r$

$$
\begin{aligned}
A &= \int_D dV = \int_0^{2\pi} d\theta \int_0^{1+\cos\theta} dr\ r \\
&= \int_0^{2\pi} d\theta\ \frac{1}{2}(1+\cos\theta)^2 \\
&= \frac{1}{2}\int_0^{2\pi} d\theta\ (1+2\cos\theta+\cos^2\theta) \\
&= \frac{1}{2}(2\pi+0+\pi) = \frac{3}{2}\pi, \qquad (37)
\end{aligned}
$$

since $\cos\theta$ integrates to zero and the average value of $\cos^2\theta$ is $\frac{1}{2}$.

Similarily

$$\begin{aligned}\int_D xdV &= \int_0^{2\pi} d\theta \int_0^{1+\cos\theta} dr\ r^2\cos\theta \\ &= \frac{1}{3}\int_0^{2\pi} d\theta\ \frac{1}{2}(1+\cos\theta)^3\cos\theta \\ &= \frac{1}{3}\int_0^{2\pi} d\theta\ (\cos\theta + 3\cos^2\theta + 3\cos^3\theta + \cos^4\theta) \\ &= \frac{1}{3}\left(3\pi + 0 + \frac{3}{4}\pi\right) = \frac{5}{4}\pi, \end{aligned} \tag{38}$$

and so $\bar{x} = 5/6$. By symmetry $\bar{y} = 0$.

4. Check that the Jacobian for the transformation from cartesian to spherical polar coordinates is

$$J = r^2 \sin\theta.$$

Consider the hemisphere defined by

$$\sqrt{x^2+y^2+z^2} \le 1, \qquad z \ge 0.$$

Using spherical polar coordinates compute its volume and centroid.

*Solution:*Spherical polar coordinates are defined by

$$\begin{aligned} x &= r\sin\theta\cos\phi, \\ y &= r\sin\theta\sin\phi, \\ z &= r\cos\theta. \end{aligned} \tag{39}$$

The Jacobian is

$$\begin{aligned}\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} &= \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}. \\ &= r^2\sin^2\theta\left[\cos^2\theta\cos^2\phi + \cos^2\theta^2\sin^2\phi^2 + \sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi\right] \\ &= r^2\sin\theta. \end{aligned} \tag{40}$$

Volume $= \int_D dV$. Centroid $\bar{x} = \bar{y} = 0$ by symmetry and $\bar{z} = \int_D dV\ z / \int_D dV$. Now

$$\begin{aligned}\int_D dV &= \int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta \int_0^1 dr\ r^2\sin\theta = 2\pi\int_0^{\pi/2} d\theta\ \sin\theta\ \frac{1}{3}. \\ &= -\frac{2}{3}\pi\cos\theta\Big|_0^{\pi/2} = 2\pi/3 \end{aligned} \tag{41}$$

as expected.

The other integral is

$$\begin{aligned}\int_D dV\ z &= \int_0^{2\pi} d\phi \int_0^{\frac{1}{2}\pi}\ d\theta\ \int_0^1\ dr\ r^2 \sin\theta \cdot r\cos\theta = 2\pi \int_0^{\frac{1}{2}\pi}\ d\theta\ \sin\theta\cos\theta\ \frac{1}{4} \\ &= \frac{\pi}{2}\int_0^{\frac{1}{2}\pi}\ d\theta\ \frac{1}{2}\sin 2\theta = \frac{\pi}{4} \end{aligned} \tag{42}$$

and therefore $\bar{z} = 3/8$.