

## 231 Outline Solutions Tutorial Sheet 1, 2 and 3.<sup>1,2</sup>

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### Problem Sheet 1

1. Rewrite the integral

$$I = \int_0^1 dx \int_1^{e^x} dy \phi(x, y) \quad (1)$$

as a double integral with the opposite order of integration.

*Solution:* The range of  $y$  values:  $1 \leq y \leq e$ . For a fixed  $y$ ,  $x$  has the range  $\log y \leq x \leq 1$ . Hence

$$I = \int_1^e dy \int_{\log y}^1 dx \phi(x, y). \quad (2)$$

2. Evaluate

$$I = \int_D dx dy x e^{xy} \quad (3)$$

where  $D$  is given by  $0 < x < 1$  and  $2 < y < 4$ .

*Solution:* So rewriting as an iterated integral

$$I = \int_D dx dy x e^{xy} = \int_0^1 dx \int_2^4 dy x e^{xy} \quad (4)$$

and integrating from the middle

$$\int_0^1 dx \int_2^4 dy x e^{xy} = \int_0^1 dx \left( x \frac{1}{x} e^{xy} \right)_2^4 = \int_0^1 dx (e^{4x} - e^{2x}) = \frac{1}{4} e^4 - \frac{1}{2} e^2 + \frac{1}{4} \quad (5)$$

Here we cunningly made the integration easier by doing the  $y$  integration first, in fact it shouldn't make any difference to the answer if the integration is done in the other order, it is definitely harder though:

$$I = \int_D dx dy x e^{xy} = \int_2^4 dy \int_0^1 dx x e^{xy} = \int_2^4 dy \left( \frac{1}{y^2} - \frac{1}{y^2} e^y + \frac{1}{y} e^y \right) \quad (6)$$

where we did the  $x$  integral using integration by parts, Now integrating by parts

$$\int_2^4 dy \frac{1}{y} e^y = \frac{1}{y} e^y \Big|_2^4 + \int_2^4 dy \frac{1}{y^2} e^y \quad (7)$$

so, substituting this in and cancelling

$$\int_2^4 dy \left( \frac{1}{y^2} - \frac{1}{y^2} e^y + \frac{1}{y} e^y \right) = \frac{1}{y} e^y \Big|_2^4 + \int_2^4 dy \left( \frac{1}{y^2} \right) = \frac{1}{4} e^4 - \frac{1}{2} e^2 + \frac{1}{4} \quad (8)$$

as before.

3. Evaluate

$$I = \int_D dx dy (x + y) \quad (9)$$

where  $D$  is given by  $0 < y < 1$  and  $2y < x < 2$ .

*Solution:* So, write as an iterated integral

$$I = \int_D dx dy (x + y) = \int_0^1 dy \int_{2y}^2 dx (x + y) \quad (10)$$

and integrate from the inside out

$$\begin{aligned} \int_0^1 dy \int_{2y}^2 dx (x + y) &= \int_0^1 dy \left( \frac{1}{2} x^2 + xy \right)_{2y}^2 = \int_0^1 dy (2 + 2y - 4y^2) \\ &= 3 - \frac{4}{3} = \frac{5}{3} \end{aligned} \quad (11)$$

4. Change the order of integration of

$$I = \int_0^{1/2} dy \int_{-\sqrt{1-4y^2}}^{\sqrt{1-4y^2}} dx y \quad (12)$$

and evaluate.

*Solution:* So

$$x = \pm \sqrt{1 - 4y^2} \quad (13)$$

implies

$$y = \pm \frac{1}{2} \sqrt{1 - x^2} \quad (14)$$

and it is easy to see from drawing a picture that

$$I = \int_0^{1/2} dy \int_{-\sqrt{1-4y^2}}^{\sqrt{1-4y^2}} dx y = \int_{-1}^1 dx \int_0^{\frac{1}{2}\sqrt{1-x^2}} dy y \quad (15)$$

Now, integrating we get

$$\int_{-1}^1 dx \int_0^{\frac{1}{2}\sqrt{1-x^2}} dy y = \frac{1}{8} \int_{-1}^1 dx (1 - x^2) = \frac{1}{8} \left( 2 - \frac{2}{3} \right) = \frac{1}{6} \quad (16)$$

<sup>1</sup>Conor Houghton, [houghton@maths.tcd.ie](mailto:houghton@maths.tcd.ie), see also <http://www.maths.tcd.ie/~houghton/231>

<sup>2</sup>Including material from Chris Ford, to whom many thanks.

## Problem Sheet 2

1. Consider the integral

$$I = \int_D dV \phi \quad (17)$$

where  $D$  is the interior of the ellipsoid defined by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1. \quad (18)$$

Write down  $I$  as an iterated triple integral.

*Solution:* Upper surface of ellipsoid is

$$z = +c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}. \quad (19)$$

whereas the lower surface is

$$z = -c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}} \quad (20)$$

The surfaces join at  $z = 0$  where  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ , this provides range of  $x$  and  $y$  integrations:  $y = -b\sqrt{1 - \frac{x^2}{a^2}}$  to  $y = +b\sqrt{1 - \frac{x^2}{a^2}}$  and  $x = -a$  to  $x = a$ :

$$I = \int_{-a}^a dx \int_{-b\sqrt{1 - \frac{x^2}{a^2}}}^{+b\sqrt{1 - \frac{x^2}{a^2}}} dy \int_{-c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}}^{+c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}} dz \phi(x, y, z). \quad (21)$$

2. The Gaussian integral formula

$$\int_{-\infty}^{\infty} dx e^{-x^2} = \sqrt{\pi} \quad (22)$$

can be derived easily with the help of polar coordinates. The trick is to note that the *square* of the integral can be recast as a double integral over  $R^2$ :

$$\left( \int_{-\infty}^{\infty} dx e^{-x^2} \right)^2 = \int_{R^2} dA e^{-x^2 - y^2}. \quad (23)$$

By changing to polar coordinates evaluate this integral.

*Solution:* After changing to polars and making sure to include the Jacobian  $J = r$

$$\int_{R^2} dA e^{-x^2 - y^2} = \int_0^{2\pi} d\theta \int_0^{\infty} dr r e^{-r^2} \quad (24)$$

and then do this integral by substituting  $u = r^2$  so  $du = 2r dr$  to give

$$I^2 = \pi \int_0^{\infty} du e^{-u} = \pi \quad (25)$$

as required.

3. Compute the Jacobian of the transformation from cartesian to parabolic cylinder coordinates

$$x = \frac{1}{2}(u^2 - v^2), \quad y = uv. \quad (26)$$

*Solution:* Well

$$\begin{aligned} J &= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \\ &= \begin{vmatrix} u & -v \\ v & u \end{vmatrix} \\ &= u^2 + v^2. \end{aligned} \quad (27)$$

4. Determine the volume of the region enclosed by the cylinder  $x^2 + y^2 = 4$  and the planes  $y + z = 4$  and  $z = 0$ . *Suggestion: Use Cartesian coordinates.*

*Solution:* Range of integration:  $z = 0$  to  $z = 4 - y$ ,  $y = -\sqrt{4 - x^2}$  to  $y = +\sqrt{4 - x^2}$  and  $x = -2$  to  $x = 2$ . Thus the volume is

$$V = \int_{-2}^2 dx \int_{-\sqrt{4 - x^2}}^{+\sqrt{4 - x^2}} dy \int_0^{4 - y} dz = \int_{-2}^2 dx \int_{-\sqrt{4 - x^2}}^{+\sqrt{4 - x^2}} dy (4 - y), \quad (28)$$

the  $z$  integral being trivial. The  $y$  integral is also straightforward:

$$V = \int_{-2}^2 dx 8\sqrt{4 - x^2} = 8 \cdot 2\pi = 16\pi. \quad (29)$$

The final integral can be evaluated by elementary means: either make the standard substitution ( $x = 2 \sin \theta$ ) or simply note that the integral represents the area of a semi-circle of radius 2.

## Problem Sheet 3

1. Rewrite the integral

$$I = \int_0^1 dy \int_{\tan^{-1} y}^{\frac{\pi}{4}} dx \phi(x, y), \quad (30)$$

as an iterated double integral with the opposite order of integration. Compute the area of the region of integration.

*Solution:* Here  $x = \frac{1}{4}\pi$  is the right boundary and  $x = \tan^{-1} y$  is the left boundary. A quick sketch shows that the left boundary is also the upper boundary which can be written  $y = \tan x$ . The lower boundary is  $y = 0$  and  $0 \leq x \leq \frac{1}{4}\pi$ . Thus

$$I = \int_0^{\frac{1}{4}\pi} dx \int_0^{\tan x} dy \phi(x, y). \quad (31)$$

Area obtained by setting  $\phi(x, y) = 1$ :

$$\begin{aligned}
 A &= \int_0^{\frac{1}{4}\pi} dx \int_0^{\tan x} dy \\
 &= \int_0^{\frac{1}{4}\pi} dx \tan x = -\log(\cos x) \Big|_0^{\frac{1}{4}\pi} \\
 &= -\left(\log \frac{1}{\sqrt{2}} - \log 1\right) \\
 &= \frac{\log 2}{2}.
 \end{aligned} \tag{32}$$

2. Compute the element of area for elliptic cylinder coordinates which are defined as

$$x = a \cosh u \cos v \tag{33}$$

$$y = a \sinh u \sin v. \tag{34}$$

*Solution:*  $\delta A = J \delta u \delta v$  with

$$\begin{aligned}
 J &= \left\| \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \right\| = \left\| \begin{vmatrix} a \sinh u \cos v & -a \cosh u \sin v \\ a \cosh u \sin v & a \sinh u \cos v \end{vmatrix} \right\| \\
 &= a^2 (\sinh^2 u \cos^2 v + \cosh^2 u \sin^2 v)
 \end{aligned} \tag{35}$$

This can be simplified a bit:

$$\begin{aligned}
 J &= a^2 (\sinh^2 u \cos^2 v + \cosh^2 u \sin^2 v) = a^2 [\sinh^2 u (1 - \sin^2 v) + \cosh^2 u \sin^2 v] \\
 &= a^2 [\sinh^2 u + \sin^2 v (\cosh^2 u - \sinh^2 u)]
 \end{aligned} \tag{36}$$

Using  $\cosh^2 u - \sinh^2 u = 1$  gives  $J = a^2 (\sinh^2 u + \sin^2 v)$ .

3. Compute the area and centroid of the plane region enclosed by the cardioid  $r(\theta) = 1 + \cos \theta$  ( $r$  and  $\theta$  are polar coordinates).

*Solution:* Use polar coordinates to evaluate area integral;  $\theta$  ranges from 0 to  $2\pi$  and  $r$  ranges from 0 to  $1 + \cos \theta$  and the Jacobian is  $J = r$

$$\begin{aligned}
 A &= \int_D dV = \int_0^{2\pi} d\theta \int_0^{1+\cos \theta} dr r \\
 &= \int_0^{2\pi} d\theta \frac{1}{2} (1 + \cos \theta)^2 \\
 &= \frac{1}{2} \int_0^{2\pi} d\theta (1 + 2 \cos \theta + \cos^2 \theta) \\
 &= \frac{1}{2} (2\pi + 0 + \pi) = \frac{3}{2}\pi,
 \end{aligned} \tag{37}$$

since  $\cos \theta$  integrates to zero and the average value of  $\cos^2 \theta$  is  $\frac{1}{2}$ .

Similarly

$$\begin{aligned}
 \int_D x dV &= \int_0^{2\pi} d\theta \int_0^{1+\cos \theta} dr r^2 \cos \theta \\
 &= \frac{1}{3} \int_0^{2\pi} d\theta \frac{1}{2} (1 + \cos \theta)^3 \cos \theta \\
 &= \frac{1}{3} \int_0^{2\pi} d\theta (\cos \theta + 3 \cos^2 \theta + 3 \cos^3 \theta + \cos^4 \theta) \\
 &= \frac{1}{3} \left( 3\pi + 0 + \frac{3}{4}\pi \right) = \frac{5}{4}\pi,
 \end{aligned} \tag{38}$$

and so  $\bar{x} = 5/6$ . By symmetry  $\bar{y} = 0$ .

4. Check that the Jacobian for the transformation from cartesian to spherical polar coordinates is

$$J = r^2 \sin \theta.$$

Consider the hemisphere defined by

$$\sqrt{x^2 + y^2 + z^2} \leq 1, \quad z \geq 0.$$

Using spherical polar coordinates compute its volume and centroid.

*Solution:* Spherical polar coordinates are defined by

$$\begin{aligned}
 x &= r \sin \theta \cos \phi, \\
 y &= r \sin \theta \sin \phi, \\
 z &= r \cos \theta.
 \end{aligned} \tag{39}$$

The Jacobian is

$$\begin{aligned}
 \frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} &= \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin \theta \cos \phi & r \cos \theta \cos \phi & -r \sin \theta \sin \phi \\ \sin \theta \sin \phi & r \cos \theta \sin \phi & r \sin \theta \cos \phi \\ \cos \theta & -r \sin \theta & 0 \end{vmatrix} \\
 &= r^2 \sin^2 \theta [\cos^2 \theta \cos^2 \phi + \cos^2 \theta \sin^2 \phi + \sin^2 \theta \cos^2 \phi + \sin^2 \theta \sin^2 \phi] \\
 &= r^2 \sin \theta.
 \end{aligned} \tag{40}$$

Volume =  $\int_D dV$ . Centroid  $\bar{x} = \bar{y} = 0$  by symmetry and  $\bar{z} = \int_D dV z / \int_D dV$ . Now

$$\begin{aligned}
 \int_D dV &= \int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta \int_0^1 dr r^2 \sin \theta = 2\pi \int_0^{\pi/2} d\theta \sin \theta \frac{1}{3} \\
 &= -\frac{2}{3}\pi \cos \theta \Big|_0^{\pi/2} = 2\pi/3
 \end{aligned} \tag{41}$$

as expected.

The other integral is

$$\begin{aligned}
\int_D dV \, z &= \int_0^{2\pi} d\phi \int_0^{\frac{1}{2}\pi} d\theta \int_0^1 dr \, r^2 \sin \theta \cdot r \cos \theta = 2\pi \int_0^{\frac{1}{2}\pi} d\theta \sin \theta \cos \theta \frac{1}{4} \\
&= \frac{\pi}{2} \int_0^{\frac{1}{2}\pi} d\theta \frac{1}{2} \sin 2\theta = \frac{\pi}{4}
\end{aligned} \tag{42}$$

and therefore  $\bar{z} = 3/8$ .