231 Outline Solutions Tutorial Sheet 10, 11 and 12.[1][2]

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Problem Sheet 10

1. Find the Fourier series representation of the sawtooth function f defined by $f(x) = x$ for $-\pi < x < \pi$ and $f(x + 2\pi) = f(x)$.

   Solution: f is odd so $a_n = 0$ for all n.

   $$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} dx \ x \ \sin nx = -\frac{x \cos nx}{n\pi}\Big|_{\pi}^{\pi} + \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{\cos nx}{n}.$$

   The integral on the RHS is zero since it is just a cosine integrated over a full period (or n periods). Thus $b_n = -2\cos(n\pi)/n = -2(-1)^n/n$ which gives

   $$f(x) = -2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin nx.$$

2. Establish that

   $$\int_{-\pi}^{\pi} dx \ \sin mx \ \sin nx = \int_{-\pi}^{\pi} dx \ \cos mx \ \cos nx = 0,$$

   if $m \neq n$ (both m and n are integers).

   Solution: In this question m and n will be taken as positive integers. The problem can be tackled using complex exponentials or trig identities. Using the identity

   $$2 \sin A \sin B = \cos(A - B) - \cos(A + B),$$

   $$\int_{-\pi}^{\pi} dx \ \sin mx \sin nx = \frac{1}{2} \int_{-\pi}^{\pi} dx \ [\cos(m - n)x - \cos(m + n)x],$$

   which is zero (integral of cosine over full periods) provided $m - n$ and $m + n$ are non-zero. To show that

   $$\int_{-\pi}^{\pi} dx \ \cos mx \cos nx = 0,$$

   use

   $$2 \cos A \cos B = \cos(A + B) + \cos(A - B).$$

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   [1] Conor Houghton, houghton@maths.tcd.ie, see also http://www.maths.tcd.ie/~houghton/231
   [2] Including material from Chris Ford, to whom many thanks.

3. The periodic function f is defined by

   $$f(x) = \begin{cases} \sin x & 0 < x < \pi \\ 0 & -\pi < x < 0 \end{cases}$$

   and $f(x + 2\pi) = f(x)$.

   (a) Represent $f(x)$ as a Fourier series.

   Solution: This function is neither odd nor even, though the only non-zero b_n coefficient is $b_1 = \frac{1}{2}$ (since $f(x) = \frac{1}{2}\sin x + |\sin x|$ and $|\sin x|$ is even). Now to the a_n coefficients

   $$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} dx \ \cos nx \ f(x) = \frac{1}{\pi} \int_{0}^{\pi} dx \ \cos nx \ \sin x$$

   This can be computed via complex exponentials or through the identity $2 \sin A \cos B = \sin(A + B) + \sin(A - B)$:

   $$a_n = \frac{1}{2\pi} \int_{0}^{\pi} dx \ [\sin(1 + n)x + \sin(1 - n)x] = -\frac{1}{2\pi} \left(\frac{\cos(1 + n)x}{1 + n} + \frac{\cos(1 - n)x}{1 - n} \right)\Big|_{0}^{\pi}.$$

   Now $\cos(1 + n)\pi = \cos(1 - n)\pi = -(-1)^n$, and so

   $$a_n = -\frac{1}{2\pi}(-(-1)^n - 1)\left(\frac{1}{1 + n} + \frac{1}{1 + n} \right) = \frac{1}{\pi}(1 + (-1)^n)\frac{1}{1 - n^2}.$$

   This is ambiguous for $n = 1$; it is trivial to check that $a_1 = 0$. Putting everything together

   $$f(x) = \frac{1}{\pi} + \frac{2}{\pi} \sum_{n>0,\text{even}} \frac{\cos nx}{1 - n^2} + \frac{1}{2} \sin x,$$

   or

   $$f(x) = \frac{1}{\pi} + \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{\cos 2mx}{1 - 4m^2} + \frac{1}{2} \sin x.$$

   (b) Derive the remarkable formula

   $$\frac{1}{2^2 - 1} + \frac{1}{4^2 - 1} + \frac{1}{6^2 - 1} + \ldots = \frac{1}{2}.$$

   Solution: $f(0) = 0$ leads to the amazing formula

   $$\frac{1}{2^2 - 1} + \frac{1}{4^2 - 1} + \frac{1}{6^2 - 1} + \ldots = \frac{1}{2}.$$

$$= \frac{2}{\pi}\left(\left.\frac{x\sin nx}{n}\right|_0^\pi - \int_0^\pi dx\ \frac{\sin nx}{n}\right)$$

$$= 0 + \left.\frac{2}{\pi}\frac{\cos nx}{n^2}\right|_0^\pi = \frac{2}{\pi}\frac{((-1)^n - 1)}{n^2}.$$

Thus $a_n = 0$ if n is even and $a_n = -4/(\pi n^2)$ if n odd.

3. Use the Fourier series given in question 2 to compute the following sums

$$S_1 = 1 - \frac{1}{3^2} - \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} - \frac{1}{11^2} - \frac{1}{13^2} + \ldots$$

$$S_2 = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \ldots$$

Remark: With calculations of this kind it makes sense to try a quick numerical check of your answer.

Solution: To compute S_1 set $x = \pi/4$ in the Fourier series quoted in question 1

$$f\left(\frac{\pi}{4}\right) = -\frac{4}{\pi}\frac{1}{\sqrt{2}}S_1.$$

Since $f(\frac{\pi}{4}) = -\frac{\pi}{4}$ one has

$$-\frac{\pi}{4} = -\frac{1}{\sqrt{2}}\frac{4}{\pi}S_1,$$

so that

$$S_1 = \frac{\sqrt{2}\pi^2}{16}.$$

ii) The average value of $|f|^2$ is

$$\frac{1}{2\pi}\int_{-\pi}^\pi dx\ |f(x)|^2 = \frac{1}{2\pi}\int_{-\pi}^\pi dx\ \left(|x| - \frac{1}{2}\pi\right)^2 = \frac{1}{\pi}\int_0^\pi dx\ \left(x - \frac{1}{2}\pi\right)^2.$$

A short calculation gives that this is equal to $\pi^2/12$. Applying Parseval's theorem

$$\frac{\pi^2}{12} = \frac{1}{4}|a_0|^2 + \frac{1}{2}\sum_{n=1}^\infty \left(|a_n|^2 + |b_n|^2\right) = \frac{1}{2}\cdot\frac{16}{\pi^2}S_2,$$

and so

$$S_2 = \frac{\pi^4}{96}.$$

Problem Sheet 12

1. The Riemann zeta function is defined as follows

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad (s > 1).$$

(a) By applying Parseval's theorem for Fourier series to the sawtooth $f(x) = x$ for $-\pi < x < \pi$ compute

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Solution: From earlier calculations, the Fourier coeficients for the sawtooth are $a_n = 0$ and $b_n = -2(-1)^n/n$. Applying Parsevals theorem:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} dx \ x^2 = \frac{1}{2} \sum_{n=1}^{\infty} \frac{4}{n^2} = 2\zeta(2)$$

The LHS is just $\pi^2/3$ which gives $\zeta(2) = \pi^2/6$.

(b) Consider the Fourier expansion of $f(x) = x^2$, $\quad -\pi < x < \pi$, and use the result to show that

$$\zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

Solution: Consider $f(x) = x^2 \quad -\pi < x < \pi$ an even function so that $b_n = 0$. The a_n can be obtained in the usual way (although one must integrate by parts twice). An alternative way is to integrate the Fourier series for the sawtooth

$$x = -2 \sum_{n=1}^{\infty} \frac{(-1)^n \sin nx}{n} \qquad -\pi < x < \pi$$

Integration with respect to x yields

$$\frac{x^2}{2} = 2 \sum_{n=1}^{\infty} \frac{(-1)^n \cos nx}{n^2} + C,$$

where C is a constant of integration. This constant can be determined by integrating both sides from $x = -\pi$ to $x = \pi$:

$$\left. \frac{x^3}{6} \right|_{-\pi}^{\pi} = 2\pi C,$$

which gives $C = \pi^2/6$. According to Parseval's theorem the average value of $|f(x)|^2$ is given by the sum

$$\frac{1}{4}|a_0|^2 + \frac{1}{2} \sum_{n=1}^{\infty} \left(|a_n|^2 + |b_n|^2 \right),$$

where a_n and b_n are the Fourier coefficients of f. For $f(x) = x^2$, $a_n = 4(-1)^n/n^2$ for $n > 0$ and $a_0 = 4C = 2\pi^2/3$. The average value of $|f(x)|^2 = x^4$ is given by

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} x^4 \ dx = \frac{\pi^4}{5}.$$

Applying Parseval's theorem

$$\frac{\pi^4}{5} = \frac{\pi^4}{9} + 8\zeta(4),$$

and so

$$\zeta(4) = \frac{\pi^4}{8} \left(\frac{1}{5} - \frac{1}{9} \right) = \frac{\pi^4}{90}.$$

2. Compute the Fourier transform of $f(x) = e^{-a|x|}$ where a is a positive constant. Use the result to show that

$$\int_{-\infty}^{\infty} dp \ \frac{\cos p}{1 + p^2} = \frac{\pi}{e}.$$

Solution:

$$\begin{aligned}
\tilde{f}(k) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \ e^{-ikx} \ f(x) \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \ e^{-ikx} \ e^{-a|x|} = \frac{1}{2\pi} \left[\int_{0}^{\infty} dx \ e^{-ikx-ax} + \int_{-\infty}^{0} dx \ e^{-ikx+ax} \right] \\
&= \frac{1}{2\pi} \left[\left. -\frac{e^{-x(a+ik)}}{a+ik} \right|_{0}^{\infty} - \left. \frac{e^{x(a-ik)}}{a-ik} \right|_{-\infty}^{0} \right] \\
&= \frac{1}{2\pi} \left[\frac{1}{a+ik} + \frac{1}{a-ik} \right] = \frac{1}{\pi} \frac{a}{a^2 + k^2}.
\end{aligned}$$

f can be represented as a Fourier integral

$$f(x) = \int_{-\infty}^{\infty} dk \ e^{ikx} \ \tilde{f}(k) = \frac{a}{\pi} \int_{-\infty}^{\infty} dk \ \frac{e^{ikx}}{a^2 + k^2}.$$

Setting $a = 1$ and $x = 1$ gives

$$e^{-1} = \frac{1}{\pi} \int_{-\infty}^{\infty} dk \ \frac{e^{ik}}{1 + k^2}.$$

Taking the real part (and multiplying by π)

$$\frac{\pi}{e} = \int_{-\infty}^{\infty} dk \ \frac{\cos k}{1 + k^2}.$$

3. Determine the Fourier transform of the Gaussian function

$$f(x) = e^{-\alpha x^2},$$

where α is a positive constant.

Solution: Completing the square

$$\int_{-\infty}^{\infty} dx\ e^{-\alpha x^2 + \beta x} = \int_{-\infty}^{\infty} dx\ e^{-\alpha(x - \frac{\beta}{2\alpha})^2 + \frac{1}{4}\beta^2/\alpha}.$$

Making the change of variables $y = x - \frac{\beta}{2\alpha}$ gives

$$\int_{-\infty}^{\infty} dx\ e^{-\alpha x^2 + \beta x} = e^{\frac{1}{4}\beta^2/\alpha} \int_{-\infty}^{\infty} dy\ e^{-\alpha y^2} = e^{\frac{1}{4}\beta^2/\alpha} \sqrt{\frac{\pi}{\alpha}},$$

using the standard Gaussian integral formula. This derivation assumes that β is real. However, we assume the result is valid for complex β to compute the Fourier transform of $f(x) = e^{-\alpha x^2}$:

$$\tilde{f}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx\ e^{-ikx} e^{-\alpha x^2} = \frac{1}{2\pi} \cdot \sqrt{\frac{\pi}{\alpha}} e^{-\frac{1}{4}k^2/\alpha},$$

by formally taking $\beta = -ik$.