231 Tutorial Sheet 9 due: Friday 19 January[1][2]

12 January 2007

Useful facts:

- The Gauss Divergence Theorem: If D is connected region in $\mathbf{R}^3$ with a piecewise smooth surface S oriented to point out of D and if $\mathbf{F}$ is a vector field defined in a region containing D and with continuous derivatives then

$$\int_D dV \nabla \cdot \mathbf{F} = \int_S \mathbf{F} \cdot \mathbf{dA} \tag{1}$$

- The vector potential: on a domain with no internal boundary, a smooth vector vector field $\mathbf{F}$ with zero divergence has a vector field $\mathbf{A}$ so that $\mathbf{F} = \nabla \times \mathbf{A}$. For star-shapped regions, using the obivious notation

$$\mathbf{A}(\mathbf{r}) = \int_0^1 dt \mathbf{F}(t\mathbf{r}) \times t\mathbf{r} \tag{2}$$

- Hodge decomposition: under some conditions, a general smooth vector field can be written in the form $\mathbf{F} = \nabla \times \mathbf{A} + \nabla\phi$ for some smooth vector field $\mathbf{A}$ and some scalar field ϕ.

Questions

1. Using Gauss' theorem or otherwise compute the flux of the vector field $\mathbf{F} = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$ through the hemisphere $x^2 + y^2 + z^2 = 1$, $z \geq 0$ with the orientation taken upwards. What is the flux out of the whole sphere?

2. Consider, again, the vector field

$$\mathbf{F} = \frac{\mathbf{r}}{r^3}, \qquad \mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

   (a) Compute the flux of $\mathbf{F}$ out of a sphere of radius a centred at the origin.
   (b) Compute the flux of $\mathbf{F}$ out of the box $1 \leq x \leq 2$, $0 \leq y \leq 1$, $0 \leq z \leq 1$.
   (c) Compute the flux of $\mathbf{F}$ out of the box $-1 \leq x \leq 1$, $-1 \leq y \leq 1$, $-1 \leq z \leq 1$.

3. Obtain a vector potential for the solenoidal vector field: $\mathbf{F} = x\mathbf{i} + y\mathbf{j} - 2z\mathbf{k}$

4. Obtain a vector potential for the solenoidal vector field: $\mathbf{F} = e^x\mathbf{k}$.

5. Find a Hodge decomposition for the vector field $\mathbf{F} = -y\mathbf{i} + x\mathbf{j} + z\mathbf{k}$.

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[2]Including material from Chris Ford, to whom many thanks.