231 Tutorial Sheet 7¹²

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Useful facts:

• To evaluate the line integral for a parameterized curve:

$$\int_{c} \mathbf{F} \cdot \mathbf{dl} = \int_{t_1}^{t_2} \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt \tag{1}$$

where t_1 and t_2 are the parameter values corresponding to the beginnig and end of the curve.

• To evaluate the surface integral for a parameterized surface:

$$\int \int_{S} \mathbf{F} \cdot \mathbf{dA} = \int \int_{S} \mathbf{F} \cdot \left(\frac{d\mathbf{r}}{du} \times \frac{d\mathbf{r}}{dv}\right) du dv \tag{2}$$

• Stokes' theorem: for an orientable piecewise smooth surface S with an orientable piecewise smooth boundy C oriented so that $\mathbf{N} \times \mathbf{dl}$ points into S with \mathbf{n} the normal and \mathbf{dl} a tangent to C and \mathbf{F} a vector field defined in a region containing S, then

$$\int_{S} \operatorname{curl} \mathbf{F} \cdot \mathbf{dA} = \int_{C} \mathbf{F} \cdot \mathbf{dl}$$
(3)

- If S is a surface δS is its bounding curve with the orientation describe above.
- Green's theorem on the plane: led D be a region in the xy-plane bounded by a piecewise continuous curve C, if f(x, y) and g(x, y) have continuous first derivatives

$$\int_{D} \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx dy = \int_{C} \left(f dx + g dy \right) \tag{4}$$

Questions

1. Which of the following vector fields are conservative?

(a)
$$\mathbf{F} = -yz\sin x \mathbf{i} + z\cos x \mathbf{j} + y\cos x \mathbf{k}.$$

(b)
$$\mathbf{F} = \frac{1}{2}y \mathbf{i} - \frac{1}{2}x \mathbf{j}.$$

- (c) $\mathbf{F} = \frac{1}{2} (\mathbf{B} \times \mathbf{r})$ where **B** is a constant vector.
- 2. Compute the flux of the vector field $\mathbf{F} = (x + x^2)\mathbf{i} + y\mathbf{j}$ out of the cylinder defined by $x^2 + y^2 = 1$ and $0 \le z \le 1$.
- 3. Find the flux of $\mathbf{F} = z^3 \mathbf{k}$ upwards through the part of the sphere $x^2 + y^2 + z^2 = a^2$ above the z = 0 plane.
- 4. Let D be a plane region with area A whose boundary is a piecewise smooth closed curve C. Use Green's theorem to prove that the centroid (\bar{x}, \bar{y}) of D is

$$\bar{x} = \frac{1}{2A} \oint_C dy \ x^2$$

$$\bar{y} = -\frac{1}{2A} \oint_C dx \ y^2.$$

$$(5)$$

Use this result to compute the centroid of a semi-circle.

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