231 Tutorial Sheet 6: due Friday November 24[1][2]

17 November 2006

Useful facts:

- To evaluate the line integral for a parameterized curve:

$$\int_c \mathbf{F}\cdot\mathbf{dl} = \int_{t_1}^{t_2} \mathbf{F}\cdot\frac{d\mathbf{r}}{dt}dt \tag{1}$$

  where t_1 and t_2 are the parameter values corresponding to the begining and end of the curve.

- A vector field $\mathbf{F}$ is **conservative** if $\mathbf{F} = \text{grad}\,\phi$ for some scalar field ϕ. ϕ is often called a **potential** for $\mathbf{F}$.

- A vector field is path-independent if its line integral between any two points is independent of the path.

- An irrotational field on a simply connected domain is conservative.

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[2]Including material from Chris Ford, to whom many thanks.

Questions

1. For each of the following vector fields compute the line integral $\oint_C \mathbf{F} \cdot \mathbf{dl}$ where C is the unit circle in the xy-plane taken anti-clockwise.

   (a) $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$

   (b) $\mathbf{F} = y\mathbf{i} - x^2y\mathbf{j}$.

2. For each of these fields determine if $\mathbf{F}$ is conservative, if it is, by integration or otherwise, find a potential: ϕ such that $\mathbf{F} = \nabla\phi$.

   (a) $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$

   (b) $\mathbf{F} = x^2y\mathbf{i} + 5xy^2\mathbf{j}$

   (c) $\mathbf{F} = e^x \cos y\mathbf{i} - e^x \sin y\mathbf{j}$

   (d) $\mathbf{F} = x \log y\mathbf{i} + y \log x\mathbf{j}$

3. If C is a straight line from (x', y, z) to (x, y, z) show

$$\int_C \mathbf{F} \cdot \mathbf{dl} = \int_{x'}^{x} F_1 dx \qquad (2)$$

4. Consider the 'point vortex' vector field

$$\mathbf{F} = \frac{y}{x^2 + y^2}\mathbf{i} - \frac{x}{x^2 + y^2}\mathbf{j}.$$

   Show that curl $\mathbf{F} = 0$ away from the z-axis. Establish that $\mathbf{F}$ is *not* conservative in the (non simply-connected) domain $x^2 + y^2 \geq \frac{1}{2}$. Is $\mathbf{F}$ conservative in the domain defined by $x^2 + y^2 \geq \frac{1}{2}$, $y \geq 0$? If so obtain a scalar potential for $\mathbf{F}$.