231 Tutorial Sheet 5[1,2]

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Useful facts:

- For a scalar field ϕ the gradiant is

$$\text{grad}\,\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k} \tag{1}$$

- For a vector field $\mathbf{F} = (F_1, F_2, F_3)$ the divergence is

$$\text{div}\,\mathbf{F} = \nabla\cdot\mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \tag{2}$$

- For a vector field $\mathbf{F} = (F_1, F_2, F_3)$ the curl is

$$\text{curl}\,\mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \tag{3}$$

- The Laplacian:

$$\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{4}$$

- To evaluate the line integral for a parameterized curve:

$$\int_c \mathbf{F}\cdot d\mathbf{l} = \int_{t_1}^{t_2} \mathbf{F}\cdot\frac{d\mathbf{r}}{dt}dt \tag{5}$$

  where t_1 and t_2 are the parameter values corresponding to the begining and end of the curve.

- A vector field is *solinoidal* if it has zero divergence.
- A vector field is *irrotational* if it has zero curl.

[1]Conor Houghton, `houghton@maths.tcd.ie`, see also `http://www.maths.tcd.ie/~houghton/231`
[2]Including material from Chris Ford, to whom many thanks.

Questions

1. Show that away from the origin the vector field

$$\mathbf{F} = \frac{\hat{\mathbf{r}}}{r^2} = \frac{\mathbf{r}}{r^3} \tag{6}$$

   is irrotational (here $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $r = |\mathbf{r}| = \sqrt{x^2+y^2+z^2}$).

2. Calculate curl $\mathbf{r}/r$ and div $\mathbf{r}/r$ away from the origin. What is $\triangle r$?

3. Prove the identity

$$\nabla\cdot(\nabla\times\mathbf{F}) = 0 \tag{7}$$

4. Prove the identity

$$\nabla\times(\nabla\times\mathbf{F}) = \nabla(\nabla\cdot\mathbf{F}) - \triangle\mathbf{F}. \tag{8}$$

5. Compute the line integrals:

   (a) $\int_C (dx\ xy\ + \frac{1}{2}dy\ x^2\ + dz)$ where C is the line segment joining the origin and the point $(1, 1, 2)$.

   (b) $\int_C (dx\ yz\ + dy\ xz\ + dz\ yx^2)$ where C is the same line as in the previous part