231 Tutorial Sheet 3: due Friday November 3[1][2]

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Useful facts:

- The Jacobian in three-dimensions:

$$dx_1 dx_2 d_3 = J dy_1 dy_2 dy_3 \tag{1}$$

where

$$J = \frac{\partial(x_1, x_2, x_3)}{\partial(y_1, y_2, y_3)} = \left\| \begin{matrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \frac{\partial x_1}{\partial y_3} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \frac{\partial x_2}{\partial y_3} \\ \frac{\partial x_3}{\partial y_1} & \frac{\partial x_3}{\partial y_2} & \frac{\partial x_3}{\partial y_3} \end{matrix} \right\| \tag{2}$$

- Some trigonometric integrals are required. In particular you may quote the integrals:

$$\begin{aligned} \int_0^{2\pi} d\theta \; \cos\theta &= 0, \\ \int_0^{2\pi} d\theta \; \cos^2\theta &= \pi, \\ \int_0^{2\pi} d\theta \; \cos^3\theta &= 0 \\ \int_0^{2\pi} d\theta \; \cos^4\theta &= \frac{3}{4}\pi \end{aligned} \tag{3}$$

Two are zero by symmetry, the other two can be computed through standard trigonometric identities or via complex exponentials:

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}). \tag{4}$$

- For a scalar field ϕ the gradiant is

$$\text{grad}\,\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k} \tag{5}$$

- If $\mathbf{a}$ is a vector $\hat{\mathbf{a}}$ is the corresponding unit vector

$$\hat{\mathbf{a}} = \frac{1}{|\mathbf{a}|}\mathbf{a} \tag{6}$$

- The direction derivative of a scalar field f in the $\mathbf{a}$ direction is $D_{\mathbf{a}} f = \hat{\mathbf{a}} \cdot \nabla f$.

[1] Conor Houghton, houghton@maths.tcd.ie, see also http://www.maths.tcd.ie/~houghton/231

[2] Including material from Chris Ford, to whom many thanks.

Questions

1. Rewrite the integral

$$I = \int_0^1 dy \int_{\tan^{-1} y}^{\frac{\pi}{4}} dx \ \phi(x, y), \tag{7}$$

as an iterated double integral with the opposite order of integration. Compute the area of the region of integration.

2. Compute the element of area for elliptic cylinder coordinates which are defined as

$$x = a \cosh u \cos v \tag{8}$$
$$y = a \sinh u \sin v. \tag{9}$$

3. Compute the area and centroid of the plane region enclosed by the cardioid $r(\theta) = 1 + \cos\theta$ (r and θ are polar coordinates).

4. Check that the Jacobian for the transformation from cartesian to spherical polar coordinates is

$$J = r^2 \sin\theta. \tag{10}$$

Consider the hemisphere defined by

$$\begin{aligned} \sqrt{x^2 + y^2 + z^2} &\leq 1 \\ z &\geq 0 \end{aligned} \tag{11}$$

Using spherical polar coordinates compute its volume and centroid.