231 Tutorial Sheet 19.[1][2]

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Useful facts:

- Series solution: assume there is a solution of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

and, by substituting into the equation find a recursion relation: an equation relating higher terms in a_n to lower one.

- By expanding out the sum it is easy to see $y(0) = a_0$ and $y'(0) = a_1$

- The method of Froebenius: assume there is a solution of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+s}$$

under the assumption that $a_0 \neq 0$, this assumption will give an equation for s called the indicial equation.

- If $y = u$ solves $y'' + py' + qy = 0$ then a second solution is given by $y = uv$ where $v'' + (p + 2\log u')v' = 0$.

[1]Conor Houghton, houghton@maths.tcd.ie, see also http://www.maths.tcd.ie/~houghton/231
[2]Including material from Chris Ford, to whom many thanks.

Questions

1. Bessel's equation reads
$$x^2y'' + xy' + (x^2 - \nu^2)y = 0.$$
In the lectures it was shown that inserting a Frobenius series of the form
$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+s}$$
with $a_0 \neq 0$ leads to $a_1 = 0$, the indical equation, $s^2 - \nu^2 = 0$, and the recursion relation
$$a_{n+2} = -\frac{a_n}{(n+s+2)^2 - \nu^2}.$$
For $\nu = 0$ this leads to two equal roots $s = 0$ and so the method only provides one solution. Use the recursion relation to compute the a_n for this case.

2. Use the method of Frobenius to obtain the general solution to the ODE
$$4xy''(x) + 2y'(x) + y(x) = 0.$$

3. In each of the following cases find a second solution in the form $y(x) = u(x)v(x)$ where $u(x)$ is a solution and $v(x)$ is to be determined.
   (a) $y'' + 6y' + 5y = 0$; one solution is $u(x) = e^{-2x}$.
   (b) $(1 - x^2)y'' - 2xy' = 0$ one solution is $u(x) = 1$.

   Remark: Part b) is the $\alpha = 0$ case of Legendre's equation.

4. The Legendre polynomials $P_n(x)$ are generated by
$$\Phi(x, h) = \frac{1}{\sqrt{1 - 2xh + h^2}} = \sum_{n=0}^{\infty} h^n P_n(x) \tag{1}$$
Write down the first four Legendre polynomials and verify that they are orthogonal
$$\int_{-1}^{1} P_n(x)P_m(x)dx = 0 \tag{2}$$
for $n \neq m$.