231 Tutorial Sheet 18: due Friday 20 April[1,2]

13 April 2007

Useful facts:

- Series solution: assume there is a solution of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

and, by substituting into the equation find a recursion relation: an equation relating higher terms in a_n to lower one.

- By expanding out the sum it is easy to see $y(0) = a_0$ and $y'(0) = a_1$

- The method of Froebenius: assume there is a solution of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+s}$$

under the assumption that $a_0 \neq 0$, this assumption will give an equation for s called the indicial equation.

[1]Conor Houghton, houghton@maths.tcd.ie, see also http://www.maths.tcd.ie/~houghton/231
[2]Including material from Chris Ford, to whom many thanks.

Questions

1. Use the recursion relation

$$a_{n+2} = \frac{2(n-\alpha)a_n}{(n+1)(n+2)}$$

or the generating function

$$\Phi(x,h) = e^{2xh-h^2} = \sum_{n=0}^{\infty} \frac{h^n}{n!} H_n(x)$$

to obtain polynomial solutions of Hermite's equation $y'' - 2xy' + 2\alpha y = 0$ for $\alpha = 3$, 4 and 5.

2. Legendre's equation can be written

$$(1-x^2)y'' - 2xy' + \alpha y = 0,$$

where α is a constant. Consider a series solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Determine a recursion relation for the a_n coefficients. For what values of α does Legendre's equation have polynomial solutions?

3. (Frobenius training exercise) For each of the following equations obtain the indicial equation for a Frobenius series of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+s}$$

   (a) $y'' + y = 0.$

   (b) $x^2 y'' + 3xy' + y = 0$

   (c) $4xy'' + 2y' + y = 0.$

In case a) use the method of Frobenius to obtain the general solution. In case b) use the method of Frobenius to find one solution (the method fails to give the other solution).

4. Use the recursion relation to show that the functions H_n defined through the generating function

$$\Phi(x,h) = e^{2xh-h^2} = \sum_{n=0}^{\infty} \frac{h^n}{n!} H_n(x)$$

satisfy Hermites equation

$$y'' - 2xy' + 2ny = 0.$$