231 Tutorial Sheet 16.[1][2]

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Useful fact:

- To solve the equation $ay'' + by' + cy = 0$, with a, b and c constants, use an exponential substitution $y = \exp(\lambda x)$ and solve for λ. If this only gives one solution, then $y = x\exp(\lambda x)$ is also a solution.

- The general solution of the equation $ay'' + by' + cy = d\exp(\mu x)$, with a, b, c and d constants is of the form $y = y_p + y_c$ where y_c is the general solution to $ay'' + by' + cy = 0$. To find y_p, substitute $y_p = C\exp\mu x$ and solve for C. This doesn't work is $exp\mu x$ solves $ay'' + by' + cy = 0$, in which case substitute $y_p = Cx\exp\mu x$, this in turn won't work if $x\exp\mu x$ also solves $ay'' + by' + cy = 0$, in which case substitute $y_p = Cx^2\exp\mu x$.

- To solve $ay'' + by' + cy = f(x)$ where $f(x)$ isn't an exponetial, then write $f(x)$ in terms of exponentials, using Fourier methods if needed. For example, say

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

  then find a particular solution y_n to the equation

$$ay'' + by' + cy = c_n e^{inx}$$

  for all n, the general solution to $ay'' + by' + cy = f(x)$ is then

$$y = y_c + \sum_{n=-\infty}^{\infty} y_n$$

  where, again, y_c solves the corresponding homogeneous equation.

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[2]Including material from Chris Ford, to whom many thanks.

Questions

1. Obtain the general solutions of the ODEs
   (a) $y'' + 5y' + 6y = e^x$
   (b) $y'' + 5y' + 6y = e^{-2x}$
   (c) $y'' + 5y' + 6y = \sinh x$
2. Obtain the general solution of the ODE
$$y''(x) + 3y'(x) + 3y(x) = f(x)$$
   where f is the periodic function defined by $f(x) = |x| - \frac{1}{2}\pi$ for $-\pi < x < \pi$ and $f(x + 2\pi) = f(x)$.
3. Obtain the general solutions of the ODEs
   (a) $y'' + y' + 3y = 0$
   (b) $y'' + y = f(x)$, where f is the periodic square wave defined by
$$f(x) = \begin{cases} 1, & 0 < x < \pi \\ -1, & -\pi < x < 0 \end{cases} \quad \text{and } f(x + 2\pi) = f(x)$$
   (c) $y'' + y' + 3y = e^{-|x|}$.