231 Tutorial Sheet 16.[1,2]

2 March 2006

Useful fact:

- To solve the equation $ay''+by'+cy=0$, with a, b and c constants, use an exponential substitution $y = \exp(\lambda x)$ and solve for λ. If this only gives one solution, then $y = x\exp(\lambda x)$ is also a solution.

- The general solution of the equation $ay''+by'+cy = d\exp(\mu x)$, with a, b, c and d constants is of the form $y = y_p + y_c$ where y_c is the general solution to $ay''+by'+cy=0$. To find y_p, substitute $y_p = C\exp\mu x$ and solve for C. This doesn't work is $exp\mu x$ solves $ay''+by'+cy=0$, in which case substitute $y_p = Cx\exp\mu x$, this in turn won't work if $x\exp\mu x$ also solves $ay''+by'+cy=0$, in which case substitute $y_p = Cx^2\exp\mu x$.

- To solve $ay''+by'+cy=f(x)$ where $f(x)$ isn't an exponetial, then write $f(x)$ in terms of exponentials, using Fourier methods if needed. For example, say

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

then find a particular solution y_n to the equation

$$ay''+by'+cy = c_n e^{inx}$$

for all n, the general solution to $ay''+by'+cy=f(x)$ is then

$$y = y_c + \sum_{n=-\infty}^{\infty} y_n$$

where, again, y_c solves the corresponding homogeneous equation.

[1]Conor Houghton, `houghton@maths.tcd.ie`, see also `http://www.maths.tcd.ie/~houghton/231`

[2]Including material from Chris Ford, to whom many thanks.

Questions

1. Obtain the general solutions of the ODEs
   (a) $y''+5y'+6y=e^x$
   (b) $y''+5y'+6y=e^{-2x}$
   (c) $y''+5y'+6y=\sinh x$

2. Obtain the general solution of the ODE

$$y''(x)+3y'(x)+3y(x) = f(x)$$

   where f is the periodic function defined by $f(x) = |x| - \frac{1}{2}\pi$ for $-\pi < x < \pi$ and $f(x+2\pi) = f(x)$.

3. Obtain the general solutions of the ODEs
   (a) $y''+y'+3y=0$
   (b) $y''+y=f(x)$, where f is the periodic square wave defined by

$$f(x) = \begin{cases} 1, & 0 < x < \pi \\ -1, & -\pi < x < 0 \end{cases} \quad \text{and } f(x+2\pi) = f(x)$$

   (c) $y''+y'+3y=e^{-|x|}$.