231 Tutorial Sheet 13.[1,2]

9 Febuary 2007

Useful facts:

- The Fourier integral or Fourier transform:

$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty} dk\, \widetilde{f(k)} e^{ikx} \\ \widetilde{f(k)} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dx\, f(x) e^{-ikx} \end{aligned}$$

- The Dirac delta function:

$$\int_{-\infty}^{\infty} dx\, f(x)\delta(x) = f(0)$$

- If $h(x)$ is a continuous function

$$\int_{-\infty}^{\infty} dx\, f(x)\delta[h(x)] = \sum_{x_i : f(x_i)=0} \frac{f(x_i)}{|h'(x_i)|}$$

  This formula hasn't been proven yet; I will be doing it on monday. It is required for some parts of question 3.

Questions

1. Express the following functions as Fourier integrals:

   (a)
   $$f(x) = \begin{cases} \cos x & |x| < \frac{\pi}{2} \\ 0 & |x| > \frac{\pi}{2} \end{cases}$$

   (b)
   $$f(x) = \frac{\sin x}{x}$$

[1]Conor Houghton, `houghton@maths.tcd.ie`, see also `http://www.maths.tcd.ie/~houghton/231`
[2]Including material from Chris Ford, to whom many thanks.

2. Prove the following properties of the Fourier transform

   (a) The Fourier transform of an even function is even.

   (b) The Fourier transform of a real odd function is purely

   (c) $\tilde{f}'(k) = ik\tilde{f}(k)$.

   (d) Acting with the Fourier transform four times reproduces the original function apart from an overall constant.

3. Compute

   (a)
$$\int_{-\infty}^{\infty} dx\ x^2\ \delta(x-3)$$

   (b)
$$\int_{-\infty}^{\infty} dx\ \delta(x^2+x)$$

   (c)
$$\int_{0}^{2} dx\ e^x\ \delta'(x-1)$$

   (d)
$$\int_{0}^{\infty} dx\ e^{-ax}\delta(\cos x)$$

   (e)
$$\int_{0}^{\infty} dx\ \delta(e^{ax}\cos x).$$

   (f)
$$\frac{d}{dx}e^{a\theta(x)}.$$

   where a is a constant.