231 Tutorial Sheet 12: due 9 Febuary 2007[1,2]

2 Febuary 2007

Useful facts:

- Parceval's formula:

$$\frac{1}{l}\int_{-l/2}^{l/2} dx|f(x)|^2 = \frac{1}{4}a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right) = \sum_{n=-\infty}^{\infty} |c_n|^2$$

- The Fourier integral or Fourier transform:

$$f(x) = \int_{-\infty}^{\infty} dk\, \widetilde{f(k)} e^{ikx}$$

$$\widetilde{f(k)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx\, f(x) e^{-ikx}$$

Questions

1. The Riemann zeta function is defined as follows

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s} \qquad (s > 1).$$

   (a) By applying Parseval's theorem for Fourier series to the sawtooth $f(x) = x$ for $-\pi < x < \pi$ compute

$$\zeta(2) = \sum_{n=1}^{\infty}\frac{1}{n^2}.$$

   (b) Consider the Fourier expansion of $f(x) = x^2, \quad -\pi < x < \pi$, and use the result to show that

$$\zeta(4) = \sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}.$$

   Remark:

   It is straightforward to compute $\zeta(n)$ if n is an even integer. On the other hand, no simple formulae are available for the odd integers, i.e. $\zeta(3)$, $\zeta(5)$, etc. Though, of course, they can be computed numerically.

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[2]Including material from Chris Ford, to whom many thanks.

The definition of $\zeta(s)$ straightforwardly extends to complex values of s. The sum defining $\zeta(s)$ clearly converges for any complex s where Re $s > 1$. In fact, the function can be 'analytically continued' to *any* complex s (apart from $s = 1$). Such a continuation boils down to finding an alternative expression for $\zeta(s)$ that agrees with the given definition when Re$s > 1$ but makes sense for any $s \neq 1$. It is rather easy to continue to s values such that Re $s > 0$. One recasts the zeta function as an alternating series

$$\zeta(s) = \frac{1}{1 - 2^{1-s}} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^s}.$$

A long standing problem in mathematics is to find the zeros of $\zeta(s)$. It is known that $\zeta(s)$ has zeros at the points $s = -2n$ where n is a natural number and that all other zeros lie in the strip $0 \leq \text{Re } s \leq 1$.

Riemann's hypothesis: All zeros of $\zeta(s)$ in the strip $0 \leq \text{Re } s \leq 1$ lie on the line Re $s = \frac{1}{2}$.

2. Compute the Fourier transform of $f(x) = e^{-a|x|}$ where a is a positive constant. Use the result to show that

$$\int_{-\infty}^{\infty} dp \; \frac{\cos p}{1 + p^2} = \frac{\pi}{e}.$$

3. Determine the Fourier transform of the Gaussian function

$$f(x) = e^{-\alpha x^2},$$

where α is a positive constant.

Suggestions: Compute the integral

$$\int_{-\infty}^{\infty} dx \; e^{-\alpha x^2 + \beta x},$$

where β is real. To do this, first consider the case $\beta = 0$. Infer the result for $\beta \neq 0$ by a simple change of variable in the integral. Then formally allow β to be imaginary (you may wish to postpone the justification of this procedure to a later date!).