

231 Tutorial Sheet 12: due 9 February 2007¹²

2 February 2007

Useful facts:

- Parseval's formula:

$$\begin{aligned}\frac{1}{l} \int_{-l/2}^{l/2} dx |f(x)|^2 &= \frac{1}{4} a_0^2 + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \\ &= \sum_{n=-\infty}^{\infty} |c_n|^2\end{aligned}$$

- The Fourier integral or Fourier transform:

$$\begin{aligned}f(x) &= \int_{-\infty}^{\infty} dk \widetilde{f(k)} e^{ikx} \\ \widetilde{f(k)} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dx f(x) e^{-ikx}\end{aligned}$$

Questions

1. The Riemann zeta function is defined as follows

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad (s > 1).$$

- (a) By applying Parseval's theorem for Fourier series to the sawtooth $f(x) = x$ for $-\pi < x < \pi$ compute

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

- (b) Consider the Fourier expansion of $f(x) = x^2$, $-\pi < x < \pi$, and use the result to show that

$$\zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

Remark:

It is straightforward to compute $\zeta(n)$ if n is an even integer. On the other hand, no simple formulae are available for the odd integers, i.e. $\zeta(3)$, $\zeta(5)$, etc. Though, of course, they can be computed numerically.

¹Conor Houghton, houghton@maths.tcd.ie, see also <http://www.maths.tcd.ie/~houghton/231>

²Including material from Chris Ford, to whom many thanks.

The definition of $\zeta(s)$ straightforwardly extends to complex values of s . The sum defining $\zeta(s)$ clearly converges for any complex s where $\operatorname{Re} s > 1$. In fact, the function can be ‘analytically continued’ to *any* complex s (apart from $s = 1$). Such a continuation boils down to finding an alternative expression for $\zeta(s)$ that agrees with the given definition when $\operatorname{Re} s > 1$ but makes sense for any $s \neq 1$. It is rather easy to continue to s values such that $\operatorname{Re} s > 0$. One recasts the zeta function as an alternating series

$$\zeta(s) = \frac{1}{1 - 2^{1-s}} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^s}.$$

A long standing problem in mathematics is to find the zeros of $\zeta(s)$. It is known that $\zeta(s)$ has zeros at the points $s = -2n$ where n is a natural number and that all other zeros lie in the strip $0 \leq \operatorname{Re} s \leq 1$.

Riemann’s hypothesis: All zeros of $\zeta(s)$ in the strip $0 \leq \operatorname{Re} s \leq 1$ lie on the line $\operatorname{Re} s = \frac{1}{2}$.

2. Compute the Fourier transform of $f(x) = e^{-a|x|}$ where a is a positive constant. Use the result to show that

$$\int_{-\infty}^{\infty} dp \frac{\cos p}{1 + p^2} = \frac{\pi}{e}.$$

3. Determine the Fourier transform of the Gaussian function

$$f(x) = e^{-\alpha x^2},$$

where α is a positive constant.

Suggestions: Compute the integral

$$\int_{-\infty}^{\infty} dx e^{-\alpha x^2 + \beta x},$$

where β is real. To do this, first consider the case $\beta = 0$. Infer the result for $\beta \neq 0$ by a simple change of variable in the integral. Then formally allow β to be imaginary (you may wish to postpone the justification of this procedure to a later date!).