231 Tutorial Sheet 10.[1,2]

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Useful facts:

- A function $f(x)$ has period l if $f(x+l)=f(x)$, it is odd if $f(-x)=-f(x)$ and even if $f(-x)=f(x)$.
- A function with period l has the Fourier series expansion

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos\left(\frac{2\pi nx}{l}\right)+\sum_{n=1}^{\infty} b_n\sin\left(\frac{2\pi nx}{l}\right).$$

where

$$\begin{aligned} a_0 &= \frac{2}{l}\int_{-l/2}^{l/2} f(x)dx \\ a_n &= \frac{2}{l}\int_{-l/2}^{l/2} f(x)\cos\left(\frac{2\pi nx}{l}\right)dx \\ b_n &= \frac{2}{l}\int_{-l/2}^{l/2} f(x)\sin\left(\frac{2\pi nx}{l}\right)dx \end{aligned}$$

[1]Conor Houghton, houghton@maths.tcd.ie, see also http://www.maths.tcd.ie/~houghton/231

[2]Including material from Chris Ford, to whom many thanks.

Questions

1. Find the Fourier series representation of the sawtooth function f defined by $f(x)=$ for $-\pi<x<\pi$ and $f(x+2\pi)=f(x)$.
2. Establish that

$$\int_{-\pi}^{\pi} dx\ \sin mx\ \sin nx=\int_{-\pi}^{\pi} dx\ \cos mx\ \cos nx=0,$$

if $m\neq n$ (both m and n are integers).

3. The periodic function f is defined by

$$f(x)=\begin{cases}\sin x & 0<x<\pi\\ 0 & -\pi<x<0\end{cases}$$

and $f(x+2\pi)=f(x)$.

(a) Represent $f(x)$ as a Fourier series.

Remarks: This function is neither odd nor even so both sets of Fourier coeff cients are required. However it turns out that all the b_n are zero except for b Can you see why this is the case without computing an integral?

(b) Derive the remarkable formula

$$\frac{1}{2^2-1}+\frac{1}{4^2-1}+\frac{1}{6^2-1}+...=\frac{1}{2}.$$