**231 Outline Solutions Tutorial Sheet 7, 8 and 9.**[1,2]

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## Problem Sheet 7

1. Which of the following vector fields are conservative?

   (a) $\mathbf{F} = -yz\sin x\ \mathbf{i} + z\cos x\ \mathbf{j} + y\cos x\ \mathbf{k}$.

   (b) $\mathbf{F} = \frac{1}{2}y\ \mathbf{i} - \frac{1}{2}x\ \mathbf{j}$.

   (c) $\mathbf{F} = \frac{1}{2}(\mathbf{B}\times\mathbf{r})$ where $\mathbf{B}$ is a constant vector.

   *Solution:*

   (a) $\mathbf{F} = \nabla yz\cos x$ so $\mathbf{F}$ is conservative.

   (b) curl $\mathbf{F} = \mathbf{k} \neq 0$ so $\mathbf{F}$ is not conservative.

   (c) A short calculation gives curl $\mathbf{F} = \mathbf{B}$ so $\mathbf{F}$ is not conservative. Remark: $\frac{1}{2}(\mathbf{B}\times\mathbf{r})$ is a vector potential for the constant vector field $\mathbf{B}$.

2. Using Gauss' theorem or otherwise compute the flux of the vector field $\mathbf{F} = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$ through the hemisphere $x^2+y^2+z^2 = 1$, $z \geq 0$ with the orientation taken upwards. What is the flux out of the whole sphere?

   *Solution:*Let $S$ be the closed surface comprising the hemisphere $x^2+y^2+z^2 = 1$, $z \geq 0$ and the disk (needed to close the surface) $z = 0$, $x^2+y^2 \leq 1$. Using Gauss' theorem the flux of $\mathbf{F}$ out of $S$ is

$$\int_S \mathbf{F}\cdot d\mathbf{A} = \int_D \text{div }\mathbf{F}\ dV = 3\int_D (x^2+y^2+z^2)\ dV,$$

   where $D$ is the region enclosed by $S$. This integral can be worked out through spherical polar coordinates or just by splitting $D$ into small spherical half-shells of volume $2\pi r^2\delta r$:

$$\int_S \mathbf{F}\cdot d\mathbf{A} = 2\pi\int_0^1 dr\ r^2\ 3r^2 = \frac{6\pi}{5}.$$

   Now the flux out of the disk is zero since here $\mathbf{F}$ is perpendicular to the outward normal $\mathbf{n} = -\mathbf{k}$. Thus the flux through the hemisphere is $6\pi/5$. The flux out of the whole sphere is $12\pi/5$.

3. Consider, again, the vector field

$$\mathbf{F} = \frac{\mathbf{r}}{r^3}, \qquad \mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

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[2]Including material from Chris Ford, to whom many thanks.



   (a) Compute the flux of $\mathbf{F}$ out of a sphere of radius $a$ centred at the origin.

   (b) Compute the flux of $\mathbf{F}$ out of the box $1 \leq x \leq 2$, $0 \leq y \leq 1$, $0 \leq z \leq 1$.

   (c) Compute the flux of $\mathbf{F}$ out of the box $-1 \leq x \leq 1$, $-1 \leq y \leq 1$, $-1 \leq z \leq 1$.

   *Solution:*

   (a) Flux integral trivial since $\mathbf{F}\cdot\mathbf{n}$ is constant over the sphere ($\mathbf{n}$ is the outward normal). Here $\mathbf{F}\cdot\mathbf{n} = 1/a^2$. Therefore $\int_C \mathbf{F}\cdot d\mathbf{A} = 4\pi a^2\ \frac{1}{a^2} = 4\pi$. Note that this is independent of the radius of the sphere.

   (b) We know div$\mathbf{F} = 0$. Therefore Gauss' theorem implies that the flux out of the box is zero. Note that this result is very difficult to obtain via a direct surface integral computation.

   (c) Naively, Gauss' theorem also gives a zero flux in this case. However, this is not correct. Indeed, an indiscriminate use of Gauss' theorem would also give a zero flux in part i). The point is that in parts i) and iii) (but not ii) ) the 'inside' region contains the origin where $\mathbf{F}$ is singular. As in part i) the correct answer to this question is $4\pi$. This can be obtained by combining Gauss' theorem with the result of the direct calculation of part a). To do this remove a sphere (centred at the origin) of radius less than one from the box. In this region $\mathbf{F}$ is smooth and so Gauss' theorem implies that the flux out of this region is zero. However this flux comprises two parts; the flux out of the box (which we wish to compute) and the flux into the cut sphere (which from part i) is $-4\pi$). Therefore the flux out of the box must be $4\pi$.

## Problem Sheet 8

1. Obtain a vector potential for the solenoidal vector field: $\mathbf{F} = x\mathbf{i} + y\mathbf{j} - 2z\mathbf{k}$

   *Solution:* Use the formula $\mathbf{A}(\mathbf{r}) = \int_0^1 dt\ \mathbf{F}(t\mathbf{r}) \times \mathbf{r}t$. Now $\mathbf{F} = x\mathbf{i}+y\mathbf{j}-2z\mathbf{k} = \mathbf{r} - 3z\mathbf{k}$ so that

$$\mathbf{A}(\mathbf{r}) = \int_0^1 dt\ \ -3zt\mathbf{k} \times \mathbf{r}t = -3\int_0^1 dt\ t^2\ (zx\mathbf{j} - zy\mathbf{i}) = zy\mathbf{i} - zx\mathbf{j}.$$

2. Obtain a vector potential for the solenoidal vector field: $\mathbf{F} = e^x\mathbf{k}$.

   *Solution:* $\mathbf{A} = e^x\mathbf{j}$ by inspection. Using the formula actually gives a different vector potential, this is possible because the vector potential is only defined up to an irrotational field,

$$\mathbf{A}(\mathbf{r}) = \left(e^x + \frac{(1-e^x)}{x}\right)\mathbf{j} - y\left(\frac{e^x}{x} - \frac{(e^x-1)}{x^2}\right)\mathbf{i}.$$

   The two vector potentials differ by a gradient

$$\mathbf{A}_{II} - \mathbf{A}_I = \frac{1-e^x}{x}\mathbf{j} - y\left(\frac{e^x}{x} - \frac{(e^x-1)}{x^2}\right)\mathbf{i} = \nabla\phi.$$

   where

$$\phi = y\frac{1-e^x}{x}.$$

3. Find a Hodge decomposition for the vector field $\mathbf{F} = -y\mathbf{i} + x\mathbf{j} + z\mathbf{k}$.

   *Solution:* So the Hodge decomposition is $\mathbf{F} = \nabla\phi + curl\,\mathbf{A}$ which implies $\triangle\phi = 1$. A convenient choice here is

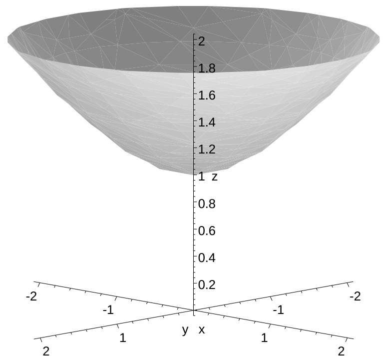
$$\phi = \frac{1}{2}z^2$$

   leaving curl$\mathbf{A} = -y\mathbf{i}+x\mathbf{j}$ but we have looked at examples like this before, $-y\mathbf{i}+x\mathbf{j} = \mathbf{k}\times\mathbf{r}$ so $\mathbf{A} = 2\mathbf{k}$.



## Problem Sheet 9

1. Using Gauss' theorem or otherwise compute the flux of the vector field $\mathbf{F} = x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ upwards through the part of the hyperboloid, $z^2 - x^2 - y^2 = 1$, lying between the $z = 1$ and $z = 2$ planes. Note that the surface to be integrated over is not a closed surface. Thus to apply Gauss' theorem consider the closed surface comprising the given surface and a suitable disc in the $z = 2$ plane. Using Gauss' theorem the flux out of the combined closed surface can be calculated. Then the disc contribution should be subtracted.




   *Solution:* Consider the closed surface comprising the given surface and the disk defined by $z = 2$ and $x^2 + y^2 \leq 3$. The flux out of this combined surface is $3V$, where $V$ is the volume enclosed, since div $\mathbf{F} = 3$.

   $V$ can be computed through cylindrical polar coordinates $(\rho, \theta, z)$

$$\int_1^2 dz \int_0^{2\pi} d\theta \int_0^{\sqrt{z^2-1}} d\rho\ \rho = \pi\int_1^2 dz\ (z^2-1) = \pi\left|\left(\frac{1}{3}z^3 - z\right)\right|_1^2 = \frac{4}{3}\pi.$$

   Thus the flux is $3V = 4\pi$.

   Now the disk contribution must be subtracted. On the disk $\mathbf{n} = \mathbf{k}$ and $\mathbf{F}\cdot\mathbf{n} = z = 2$, a constant. The flux (upwards) through the disk is $2\cdot 3\pi = 6\pi$ since the area of the disk is $3\pi$.

2. Obtain vector potentials for the solenoidal vector fields:

   (a) $\mathbf{F} = \mathbf{i} + 2z\mathbf{j} + \mathbf{k}$

   (b) $\mathbf{F} = e^z\mathbf{i} + e^x\mathbf{k}$.

   *Solution:* Use the formula $\mathbf{A}(\mathbf{r}) = \int_0^1 dt\ \mathbf{F}(t\mathbf{r}) \times \mathbf{r}t$.

(a) $\mathbf{F}(rt) = (1, 2zt, 1)$ and so the integral is

$$\mathbf{A} = \int_0^1 dt\ (t^2z^2 - yt, tx - tz, yt - xzt^2)$$

giving

$$\mathbf{A} = \left(\frac{z^2}{t} - \frac{y}{2}, \frac{x}{2} - \frac{z}{2}, \frac{y}{2} - \frac{xz}{3}\right)$$

(b) So, here,

$$\mathbf{F}(t\mathbf{r}) \times \mathbf{r}t = \left(-yte^{tx}, xte^{tx} - tze^{tz}, yte^{tz}\right)$$

and using the integration by parts formula

$$\int_0^1 te^{at}dt = \frac{(a-1)e^a + 1}{a^2}$$

gives

$$\mathbf{A} = \left(y\frac{(x-1)e^x + 1}{x^2}, \frac{(x-1)e^x + 1}{x} - \frac{(z-1)e^z + 1}{z^2}, y\frac{(z-1)e^z + 1}{z^2}\right)$$

3. Compute the lengths of the following curves

(a) A helix with parametrization $x(u) = 2\cos u$, $y(u) = 2\sin u$, $z(u) = u$ and $0 \leq u \leq 2\pi$.

(b) A lemniscate $r^2 = \cos 2\theta$ with $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$ ($r$ and $\theta$ are polar coordinates).

In part b) write the result as an integral (don't bother trying to compute it).

*Solution:*

(a) With the given parametrization

$$\frac{d\mathbf{r}}{du} = -2\sin u\ \mathbf{i} + 2\cos u\ \mathbf{j} + \mathbf{k},$$

and so

$$\left|\frac{d\mathbf{r}}{du}\right| = \sqrt{5}.$$

Therefore the arc length is $2\pi\sqrt{5}$.

(b) Here the parametrization is not given. Use the polar angle as a parameter, i.e. $u = \theta$. This gives $x = r\cos\theta = \sqrt{\cos 2u}\cos u$ and $y = \sqrt{\cos 2u}\sin u$ or $\mathbf{r}(u) = \sqrt{\cos 2u}(\cos u\ \mathbf{i} + \sin u\ \mathbf{j})$.

$$\frac{d\mathbf{r}(u)}{du} = -\frac{\sin 2u}{\sqrt{\cos 2u}}(\cos u\mathbf{i} + \sin u\mathbf{j}) + \sqrt{\cos 2u}\,(-\sin u\mathbf{i} + \cos u\mathbf{j}).$$



$$\left|\frac{d\mathbf{r}(u)}{du}\right| = \frac{1}{\sqrt{\cos 2u}}\sqrt{\sin^2 2u + \cos^2 2u} = \frac{1}{\sqrt{\cos 2u}}.$$

$$L = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{du}{\sqrt{\cos 2u}}.$$

4. Determine the surface area of the 'bowl' which is the part of the paraboloid $z = x^2 + y^2$ below the $z = 1$ plane.

*Solution:* Using the standard formula with $h(x, y) = z = x^2 + y^2$

$$A = \int_{x^2+y^2\leq 1} dx\ dy\ \sqrt{1 + \left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2} = \int_{x^2+y^2\leq 1} dx\ dy\ \sqrt{1 + 4x^2 + 4y^2}.$$

Change to polar coordinates

$$\begin{aligned} A &= \int_0^{2\pi} d\theta \int_0^1 dr\ r\ \sqrt{1 + 4r^2} \qquad \text{(Jacobian } J = r) \\ &= 2\pi\ \frac{2}{3} \cdot \frac{1}{8}\ (1 + 4r^2)^{\frac{3}{2}}\Big|_0^1 \\ &= \frac{\pi}{6}(5^{\frac{3}{2}} - 1) \end{aligned}$$

5. The centroid of a curve $C$, $(\bar{x}, \bar{y}, \bar{z})$ is given by

$$\bar{x} = \frac{1}{L}\int_C x dl \tag{1}$$

and so on, with $L$ the length of the curve. Show that the centroid of a circular arc with radius $r$ and angle $2\theta$ (at the centre) is at a distance of $r\sin\theta/\theta$ from the centre.
*Solution:* The arc can be parametrized $\mathbf{r}(u) = r\cos u\mathbf{i} + r\sin u\mathbf{j}$ where $-\theta < u < \theta$.

$$\frac{d\mathbf{r}(u)}{du} = -r\sin u\mathbf{i} + r\cos u\mathbf{j}$$

with

$$\left|\frac{d\mathbf{r}(u)}{du}\right| = r.$$

The centroid is

$$\bar{x} = \frac{1}{L}\int_{-\theta}^{\theta} du\ \left|\frac{d\mathbf{r}(u)}{du}\right| x(u) = \frac{1}{L}\int_{-\theta}^{\theta} du\ r \cdot r\cos u = \frac{2r^2\sin\theta}{L} = \frac{r\sin\theta}{\theta},$$

since $L = 2r\theta$. $\bar{y} = 0$ by symmetry and so the centroid is at a distance of $r\sin\theta/\theta$ from the centre.