231 Outline Solutions Tutorial Sheet 7, 8 and 9.[1,2]

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Problem Sheet 7

1. Which of the following vector fields are conservative?

   (a) $\mathbf{F} = -yz\sin x\ \mathbf{i} + z\cos x\ \mathbf{j} + y\cos x\ \mathbf{k}$.

   (b) $\mathbf{F} = \frac{1}{2}y\ \mathbf{i} - \frac{1}{2}x\ \mathbf{j}$.

   (c) $\mathbf{F} = \frac{1}{2}(\mathbf{B}\times\mathbf{r})$ where $\mathbf{B}$ is a constant vector.

   Solution:

   (a) $\mathbf{F} = \nabla yz\cos x$ so $\mathbf{F}$ is conservative.

   (b) curl $\mathbf{F} = \mathbf{k} \neq 0$ so $\mathbf{F}$ is not conservative.

   (c) A short calculation gives curl $\mathbf{F} = \mathbf{B}$ so $\mathbf{F}$ is not conservative. Remark: $\frac{1}{2}(\mathbf{B}\times\mathbf{r})$ is a vector potential for the constant vector field $\mathbf{B}$.

2. Using Gauss' theorem or otherwise compute the flux of the vector field $\mathbf{F} = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$ through the hemisphere $x^2+y^2+z^2 = 1$, $z \geq 0$ with the orientation taken upwards. What is the flux out of the whole sphere?

   *Solution:*Let S be the closed surface comprising the hemisphere $x^2+y^2+z^2 = 1$, $z \geq 0$ and the disk (needed to close the surface) $z = 0$, $x^2+y^2 \leq 1$. Using Gauss' theorem the flux of $\mathbf{F}$ out of S is

   $$\int_S \mathbf{F}\cdot d\mathbf{A} = \int_D \text{div}\ \mathbf{F}\ dV = 3\int_D (x^2+y^2+z^2)\ dV,$$

   where D is the region enclosed by S. This integral can be worked out through spherical polar coordinates or just by splitting D into small spherical half-shells of volume $2\pi r^2\delta r$:

   $$\int_S \mathbf{F}\cdot d\mathbf{A} = 2\pi\int_0^1 dr\ r^2\ 3r^2 = \frac{6\pi}{5}.$$

   Now the flux out of the disk is zero since here $\mathbf{F}$ is perpendicular to the outward normal $\mathbf{n} = -\mathbf{k}$. Thus the flux through the hemisphere is $6\pi/5$. The flux out of the whole sphere is $12\pi/5$.

3. Consider, again, the vector field

   $$\mathbf{F} = \frac{\mathbf{r}}{r^3}, \qquad \mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

[1]Conor Houghton, houghton@maths.tcd.ie, see also http://www.maths.tcd.ie/~houghton/231

[2]Including material from Chris Ford, to whom many thanks.

(a) Compute the flux of $\mathbf{F}$ out of a sphere of radius a centred at the origin.

(b) Compute the flux of $\mathbf{F}$ out of the box $1 \le x \le 2$, $0 \le y \le 1$, $0 \le z \le 1$.

(c) Compute the flux of $\mathbf{F}$ out of the box $-1 \le x \le 1$, $-1 \le y \le 1$, $-1 \le z \le 1$.

Solution:

(a) Flux integral trivial since $\mathbf{F} \cdot \mathbf{n}$ is constant over the sphere ($\mathbf{n}$ is the outward normal). Here $\mathbf{F} \cdot \mathbf{n} = 1/a^2$. Therefore $\int_C \mathbf{F} \cdot d\mathbf{A} = 4\pi a^2 \, \frac{1}{a^2} = 4\pi$. Note that this is independent of the radius of the sphere.

(b) We know div$\mathbf{F} = 0$. Therefore Gauss' theorem implies that the flux out of the box is zero. Note that this result is very difficult to obtain via a direct surface integral computation.

(c) Naively, Gauss' theorem also gives a zero flux in this case. However, this is not correct. Indeed, an indiscriminate use of Gauss' theorem would also give a zero flux in part i). The point is that in parts i) and iii) (but not ii)) the 'inside' region contains the origin where $\mathbf{F}$ is singular. As in part i) the correct answer to this question is 4π. This can be obtained by combining Gauss' theorem with the result of the direct calculation of part a). To do this remove a sphere (centred at the origin) of radius less than one from the box. In this region $\mathbf{F}$ is smooth and so Gauss' theorem implies that the flux out of this region is zero. However this flux comprises two parts; the flux out of the box (which we wish to compute) and the flux into the cut sphere (which from part i) is -4π). Therefore the flux out of the box must be 4π.

Problem Sheet 8

1. Obtain a vector potential for the solenoidal vector field: $\mathbf{F} = x\mathbf{i} + y\mathbf{j} - 2z\mathbf{k}$

   Solution: Use the formula $\mathbf{A}(\mathbf{r}) = \int_0^1 \, dt \, \mathbf{F}(t\mathbf{r}) \times \mathbf{r}t$. Now $\mathbf{F} = x\mathbf{i} + y\mathbf{j} - 2z\mathbf{k} = \mathbf{r} - 3z\mathbf{k}$ so that

$$\mathbf{A}(\mathbf{r}) = \int_0^1 \, dt \; -3zt\mathbf{k} \times \mathbf{r}t = -3 \int_0^1 \, dt \; t^2 \; (zx\mathbf{j} - zy\mathbf{i}) = zy\mathbf{i} - zx\mathbf{j}.$$

2. Obtain a vector potential for the solenoidal vector field: $\mathbf{F} = e^x\mathbf{k}$.

   Solution: $\mathbf{A} = e^x\mathbf{j}$ by inspection. Using the formula actually gives a different vector potential, this is possible because the vector potential is only defined up to an irrotational field,

$$\mathbf{A}(\mathbf{r}) = \left(e^x + \frac{(1-e^x)}{x}\right)\mathbf{j} - y\left(\frac{e^x}{x} - \frac{(e^x-1)}{x^2}\right)\mathbf{i}.$$

   The two vector potentials differ by a gradient

$$\mathbf{A}_{II} - \mathbf{A}_I = \frac{1-e^x}{x}\mathbf{j} - y\left(\frac{e^x}{x} - \frac{(e^x-1)}{x^2}\right)\mathbf{i} = \nabla\phi.$$

   where

$$\phi = y\frac{1-e^x}{x}.$$

3. Find a Hodge decomposition for the vector field $\mathbf{F} = -y\mathbf{i} + x\mathbf{j} + z\mathbf{k}$.

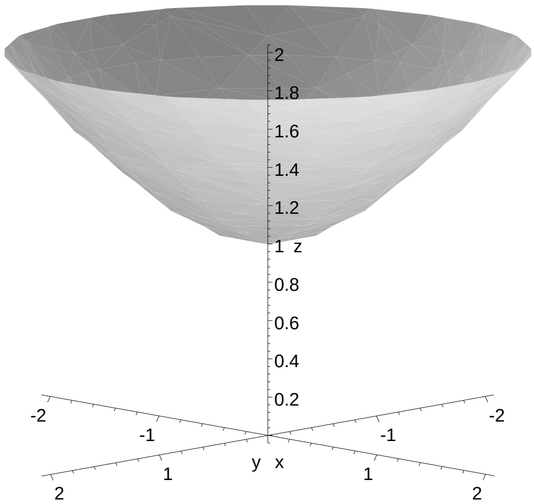
   Solution: So the Hodge decomposition is $\mathbf{F} = \nabla\phi + curl\,\mathbf{A}$ which implies $\triangle\phi = 1$. A convenient choice here is

$$\phi = \frac{1}{2}z^2$$

   leaving curl$\mathbf{A} = -y\mathbf{i} + x\mathbf{j}$ but we have looked at examples like this before, $-y\mathbf{i} + x\mathbf{j} = \mathbf{k} \times \mathbf{r}$ so $\mathbf{A} = 2\mathbf{k}$.

Problem Sheet 9

1. Using Gauss' theorem or otherwise compute the flux of the vector field $\mathbf{F} = x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ upwards through the part of the hyperboloid, $z^2 - x^2 - y^2 = 1$, lying between the $z = 1$ and $z = 2$ planes. Note that the surface to be integrated over is not a closed surface. Thus to apply Gauss' theorem consider the closed surface comprising the given surface and a suitable disc in the $z = 2$ plane. Using Gauss' theorem the flux out of the combined closed surface can be calculated. Then the disc contribution should be subtracted.



*Solution:*Consider the closed surface comprising the given surface and the disk defined by $z = 2$ and $x^2 + y^2 \leq 3$. The flux out of this combined surface is $3V$, where V is the volume enclosed, since div $\mathbf{F} = 3$.

V can be computed through cylindrical polar coordinates (ρ, θ, z)

$$\int_1^2 dz \int_0^{2\pi} d\theta \int_0^{\sqrt{z^2-1}} d\rho\ \rho = \pi \int_1^2 dz\ (z^2 - 1) = \pi \left| \left(\frac{1}{3}z^3 - z \right) \right|_1^2 = \frac{4}{3}\pi.$$

Thus the flux is $3V = 4\pi$.

Now the disk contribution must be subtracted. On the disk $\mathbf{n} = \mathbf{k}$ and $\mathbf{F} \cdot \mathbf{n} = z = 2$, a constant. The flux (upwards) through the disk is $2 \cdot 3\pi = 6\pi$ since the area of the disk is 3π.

2. Obtain vector potentials for the solenoidal vector fields:

   (a) $\mathbf{F} = \mathbf{i} + 2z\mathbf{j} + \mathbf{k}$

   (b) $\mathbf{F} = e^z\mathbf{i} + e^x\mathbf{k}$.

   *Solution:*Use the formula $\mathbf{A}(\mathbf{r}) = \int_0^1\ dt\ \mathbf{F}(t\mathbf{r}) \times \mathbf{r}t$.

(a) $\mathbf{F}(rt) = (1, 2zt, 1)$ and so the integral is

$$\mathbf{A} = \int_0^1 dt\ (t^2z^2 - yt, tx - tz, yt - xzt^2)$$

giving

$$\mathbf{A} = \left(\frac{z^2}{t} - \frac{y}{2}, \frac{x}{2} - \frac{z}{2}, \frac{y}{2} - \frac{xz}{3}\right)$$

(b) So, here,

$$\mathbf{F}(t\mathbf{r}) \times \mathbf{r}t = \left(-yte^{tx}, xte^{tx} - tze^{tz}, yte^{tz}\right)$$

and using the integration by parts formula

$$\int_0^1 te^{at}dt = \frac{(a-1)e^a + 1}{a^2}$$

gives

$$\mathbf{A} = \left(y\frac{(x-1)e^x + 1}{x^2}, \frac{(x-1)e^x + 1}{x} - \frac{(z-1)e^z + 1}{z^2}, y\frac{(z-1)e^z + 1}{z^2}\right)$$

3. Compute the lengths of the following curves

(a) A helix with parametrization $x(u) = 2\cos u$, $y(u) = 2\sin u$, $z(u) = u$ and $0 \le u \le 2\pi$.

(b) A lemniscate $r^2 = \cos 2\theta$ with $-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$ (r and θ are polar coordinates).

In part b) write the result as an integral (don't bother trying to compute it).

Solution:

(a) With the given parametrization

$$\frac{d\mathbf{r}}{du} = -2\sin u\ \mathbf{i} + 2\cos u\ \mathbf{j} + \mathbf{k},$$

and so

$$\left|\frac{d\mathbf{r}}{du}\right| = \sqrt{5}.$$

Therefore the arc length is $2\pi\sqrt{5}$.

(b) Here the parametrization is not given. Use the polar angle as a parameter, i.e. $u = \theta$. This gives $x = r\cos\theta = \sqrt{\cos 2u}\cos u$ and $y = \sqrt{\cos 2u}\sin u$ or $\mathbf{r}(u) = \sqrt{\cos 2u}(\cos u\ \mathbf{i} + \sin u\ \mathbf{j})$.

$$\frac{d\mathbf{r}(u)}{du} = -\frac{\sin 2u}{\sqrt{\cos 2u}}(\cos u\mathbf{i} + \sin u\mathbf{j}) + \sqrt{\cos 2u}\left(-\sin u\mathbf{i} + \cos u\mathbf{j}\right).$$

$$\left|\frac{d\mathbf{r}(u)}{du}\right| = \frac{1}{\sqrt{\cos 2u}}\sqrt{\sin^2 2u + \cos^2 2u} = \frac{1}{\sqrt{\cos 2u}}.$$

$$L = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{du}{\sqrt{\cos 2u}}.$$

4. Determine the surface area of the 'bowl' which is the part of the paraboloid $z = x^2+y^2$ below the $z = 1$ plane.

   *Solution:*Using the standard formula with $h(x, y) = z = x^2 + y^2$

$$A = \int_{x^2+y^2\leq 1} dx\ dy\ \sqrt{1 + \left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2} = \int_{x^2+y^2\leq 1} dx\ dy\ \sqrt{1 + 4x^2 + 4y^2}.$$

   Change to polar coordinates

$$\begin{aligned} A &= \int_0^{2\pi} d\theta \int_0^1 dr\ r\ \sqrt{1+4r^2} \qquad \text{(Jacobian } J = r\text{)} \\ &= 2\pi\ \frac{2}{3}\cdot\frac{1}{8}\ (1+4r^2)^{\frac{3}{2}}\Big|_0^1 \\ &= \frac{\pi}{6}(5^{\frac{3}{2}} - 1) \end{aligned}$$

5. The centroid of a curve C, $(\bar{x}, \bar{y}, \bar{z})$ is given by

$$\bar{x} = \frac{1}{L}\int_C x dl \tag{1}$$

   and so on, with L the length of the curve. Show that the centroid of a circular arc with radius r and angle 2θ (at the centre) is at a distance of $r\sin\theta/\theta$ from the centre.
   *Solution:*The arc can be parametrized $\mathbf{r}(u) = r\cos u\mathbf{i} + r\sin u\mathbf{j}$ where $-\theta < u < \theta$.

$$\frac{d\mathbf{r}(u)}{du} = -r\sin u\mathbf{i} + r\cos u\mathbf{j}$$

   with

$$\left|\frac{d\mathbf{r}(u)}{du}\right| = r.$$

   The centroid is

$$\bar{x} = \frac{1}{L}\int_{-\theta}^{\theta} du\ \left|\frac{d\mathbf{r}(u)}{du}\right| x(u) = \frac{1}{L}\int_{-\theta}^{\theta} du\ r\cdot r\cos u = \frac{2r^2\sin\theta}{L} = \frac{r\sin\theta}{\theta},$$

   since $L = 2r\theta$. $\bar{y} = 0$ by symmetry and so the centroid is at a distance of $r\sin\theta/\theta$ from the centre.