231 Outline Solutions Tutorial Sheet 4, 5 and 6.[1,2]

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Problem Sheet 4

1. Compute the line integrals:

   (a) $\int_C (dx\ xy\ + \frac{1}{2}dy\ x^2\ + dz)$ where C is the line segment joining the origin and the point $(1,1,2)$.

   (b) $\int_C (dx\ yz\ + dy\ xz\ + dz\ yx^2)$ where C is the same line as in the previous part

   Solution: A quick way here is to note that $\mathbf{F}$ is conservative.

   $$\mathbf{F} = xy\mathbf{i} + \frac{1}{2}x^2\mathbf{j} + \mathbf{k} = \nabla\phi \tag{1}$$

   where $\phi = \frac{1}{2}x^2y + z$. Hence

   $$\int_C \mathbf{F}\cdot d\mathbf{l} = \phi(1,1,2) - \phi(0,0,0) = \frac{5}{2}. \tag{2}$$

   For the next part, use the parametrization $x(u) = u$, $y(u) = u$, $z(u) = 2u$ $(0 \le u \le 1)$.

   $$\begin{aligned}\frac{d\mathbf{r}}{du} &= \mathbf{i}+\mathbf{j}+2\mathbf{k},\\ \mathbf{F}\cdot\frac{d\mathbf{r}}{du} &= 2u^2+2u^2+2u^3 = 4u^2+2u^3\end{aligned} \tag{3}$$

   so

   $$\int_C \mathbf{F}\cdot d\mathbf{l} = \int_0^1 du\ \left(4u^2+2u^3\right) = \frac{4}{3}+\frac{1}{2} = \frac{11}{6}. \tag{4}$$

2. Is $F = \mathbf{r}/r$ irrotational? Is it conservative. Is it conservative on a restricted domain? It is path independent?

   Solution: Well

   $$\nabla r = \mathbf{r}/r \tag{5}$$

   So this field is conservative and path independent. Interestingly, the field is not defined at $(0,0,0)$ and so this only applied on the restricted domain, however, the scalar potential is defined everywhere.

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[2] Including material from Chris Ford, to whom many thanks.

3. Consider the vector field

   $$\mathbf{F} = \frac{\frac{1}{2}y}{x^2+y^2}\mathbf{i} - \frac{\frac{1}{2}x}{x^2+y^2}\mathbf{j}. \tag{6}$$

   (a) Determine the line integral $\oint_C \mathbf{dl}\cdot\mathbf{F}$ where C is the unit circle centred at the origin in the $z = 0$ plane (taken anti-clockwise).

   (b) Compute the curl of $\mathbf{F}$.

   (c) Is $\mathbf{F}$ a conservative vector field?

   Solution: Well on the circle

   $$\mathbf{r} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j} \tag{7}$$

   and so

   $$\frac{\partial\mathbf{r}}{\partial\theta} = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j} \tag{8}$$

   and the integral become

   $$\oint \mathbf{F}\cdot\mathbf{dl} = -\int_0^{2\pi} d\theta = -2\pi \tag{9}$$

   since $\mathbf{F} = -\cos\theta\mathbf{i}+\sin\theta\mathbf{j}$ on the circle. However, the curl does vanish, for convenient write $\rho = \sqrt{x^2+y^2}$ and using the obvious notation

   $$\nabla\times\mathbf{F} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \partial_x & \partial_y & \partial_z\\ \frac{y}{2\rho^2} & -\frac{x}{2\rho^2} & 0\end{vmatrix} = \begin{pmatrix}0\\0\\ \frac{\partial}{\partial x}\frac{x}{2\rho^2} + \frac{\partial}{\partial y}\frac{y}{2\rho^2}\end{pmatrix} \tag{10}$$

   Now

   $$\frac{\partial}{\partial x}\frac{x}{2\rho^2} = \frac{1}{2\rho^2} - \frac{x^2}{\rho^4} \tag{11}$$

   and adding this to the similar result for y gives zero. So, it is not conservative, there is no contradiction here, the domain isn't simply connected, for the field to be defined the z-axis, along which $\rho = 0$ must be excluded and so for example, the unit cirle we integrated around can't be shrunk to point. We will see later, problem sheet 6, that there is locally a scalar potential, but this isn't well defined over the whole domain.

Problem Sheet 5

1. (a) Compute the flux of the vector field $\mathbf{F} = x\mathbf{i}+2y\mathbf{j}+z\mathbf{k}$ out of the cylinder defined by $x^2 + y^2 = 1$ and $0 \le z \le 1$.

   (b) What is the flux if the vector field is replaced with $\mathbf{F} = \mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$?

   (c) Find the flux of $\mathbf{F} = z^2\mathbf{k}$ upwards through the part of the sphere $x^2+y^2+z^2 = a^2$ in the first octant of $\mathbf{R}^3$ (in the first octant all three coordinates x, y and z are positive).

*Solution:*For the first part divide the cylinder up into a curved shaft and a disk at the top and bottom. For the curved part, use cylindrical polars with $\rho = 1$: $u = \theta$, $v = z$ so that $x = \cos u$, $y = \sin u$, $z = v$ or

$$\mathbf{r}(u, v) = \cos u\mathbf{i} + \sin u\mathbf{j} + v\mathbf{k} \tag{12}$$

and

$$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \cos u\mathbf{i} + \sin u\mathbf{j}. \tag{13}$$

so

$$\mathbf{F} \cdot \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = x \cos u + 2y \sin u = \cos^2 u + 2\sin^2 u. \tag{14}$$

Hence

$$\int_{\text{shaft}} \mathbf{F} \cdot \mathbf{dA} = \int_0^1 dv \int_0^{2\pi} du \,\left(\cos^2 u + 2\sin^2 u\right) = 1(\pi + 2\pi) = 3\pi. \tag{15}$$

Now, the base has $z = 0$ and $\mathbf{n} = -\mathbf{k}$ so $\mathbf{F} \cdot \mathbf{n} = 0$, the top has $z = 1$ and $\mathbf{n} = \mathbf{k}$ and hence $\mathbf{F} \cdot \mathbf{n} = 1$ so

$$\int_{\text{top}} \mathbf{F} \cdot \mathbf{dA} = \pi \tag{16}$$

because the top is a disk of radius one and hence area π. Thus

$$\int_{\text{top}} \mathbf{F} \cdot \mathbf{dA} = 4\pi \tag{17}$$

If $\mathbf{F} = \mathbf{r}$ then the integral over the shaft becomes

$$\int_{\text{shaft}} \mathbf{F} \cdot \mathbf{dA} = \int_0^1 dv \int_0^{2\pi} du \;= 2\pi. \tag{18}$$

because

$$\mathbf{r} \cdot \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \cos^2 u + \sin^2 u = 1 \tag{19}$$

and the surface area of the curved part is 2π. The flux out of the top and bottom is unchanged since only the $\mathbf{k}$ component is relavant and this is the same, hence

$$\int_{\text{top}} \mathbf{F} \cdot \mathbf{dA} = 3\pi \tag{20}$$

Finally, the octant of the sphere is parameterized by

$$\begin{aligned} x &= a\cos\phi\sin\theta \\ y &= a\sin\phi\sin\theta \\ z &= a\cos\theta \end{aligned} \tag{21}$$

and so

$$\begin{aligned} \frac{\partial \mathbf{r}}{\partial \theta} &= a\cos\phi\cos\theta\mathbf{i} + a\sin\phi\cos\theta\mathbf{j} + -a\sin\theta\mathbf{k} \\ \frac{\partial \mathbf{r}}{\partial \phi} &= -a\sin\phi\sin\theta\mathbf{i} + a\cos\phi\sin\theta\mathbf{j} \end{aligned} \tag{22}$$

and hence

$$\frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial \phi} = a\sin\theta(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \tag{23}$$

This result can also be obtained directly by noting that the area element on a sphere or radius a is $dA = a^2 \sin\theta d\theta d\phi$ and the normal is $\mathbf{r}/a$. Now

$$\mathbf{F} \cdot \mathbf{dA} = az^3 \sin\theta d\theta d\phi \tag{24}$$

and the integral becomes

$$\int_S \mathbf{F} \cdot \mathbf{dA} = a^4 \int_0^{\pi/2} d\phi \int_0^{\pi/2} d\theta \cos^3\theta \sin\theta = \frac{a^4\pi}{2} \int_0^1 \theta d\cos\theta \cos^3 = \frac{a^4\pi}{8} \tag{25}$$

2. Find the flux of $\mathbf{F} = 2x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ upwards through the surface with parametrization $\mathbf{r}(u, v) = u^2v\mathbf{i} + uv^2\mathbf{j} + v^3\mathbf{k}$ where u and v range from 0 to 1.

*Solution:*Here the parametrization of the surface is given $\mathbf{r} = u^2v\mathbf{i} + uv^2\mathbf{j} + v^3\mathbf{k}$.

$$\frac{\partial \mathbf{r}}{\partial u} = 2uv\mathbf{i} + v^2\mathbf{j}, \qquad \frac{\partial \mathbf{r}}{\partial v} = u^2\mathbf{i} + 2uv\mathbf{j} + 3v^2\mathbf{k}. \tag{26}$$

A short calculation gives

$$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = 3v^4\mathbf{i} - 6uv^3\mathbf{j} + 3u^2v^2\mathbf{k}. \tag{27}$$

Now

$$\mathbf{F} \cdot \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = 3v^4\; 2x - 6uv^3y + 3u^2v^2z = 3u^2v^5, \tag{28}$$

which gives

$$\int_S \mathbf{F} \cdot d\mathbf{A} = 3\int_0^1 du \int_0^1 dv\; u^2v^5 = \frac{1}{6}. \tag{29}$$

Problem Sheet 6

1. For each of the following vector fields compute the line integral $\oint_C \mathbf{F}\cdot d\mathbf{l}$ where C is the unit circle in the xy-plane taken anti-clockwise.

   (a) $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$

   (b) $\mathbf{F} = y\mathbf{i} - x^2y\mathbf{j}$.

   Solution: In the first part $\mathbf{F} = \nabla\frac{1}{2}(x^2+y^2)$ so that $\mathbf{F}$ is conservative giving $\oint_C \mathbf{F}\cdot d\mathbf{l}$. In the second part parametrize curve:

$$\begin{aligned} x(u) &= \cos u \\ y(u) &= \sin u \\ z(u) &= 0 \end{aligned} \tag{30}$$

   where $0 \le u \le 2\pi$ or $\mathbf{r}(u) = \cos u\mathbf{i} + \sin u\mathbf{j}$. Now

$$\frac{d\mathbf{r}(u)}{du} = -\sin u\mathbf{i} + \cos u\mathbf{j}. \tag{31}$$

   and

$$\mathbf{F}\cdot\frac{d\mathbf{r}}{du} = -y\sin u - x^2y\cos u = -\sin^2 u - \cos^3 u\sin u. \tag{32}$$

   Thus

$$\oint_C \mathbf{F}\cdot d\mathbf{l} = \int_0^{2\pi} du\ \left(-\sin^2 u - \cos^3 u\sin u\right) = -\pi, \tag{33}$$

   since the average value of $\sin^2 u$ is $\frac{1}{2}$ and $\int_0^{2\pi} du\ \cos^3 u\sin u = 0$ by symmetry.

2. Compute the flux of the vector field $\mathbf{F} = (x+x^2)\mathbf{i} + y\mathbf{j}$ out of the cylinder defined by $x^2+y^2=1$ and $0 \le z \le 1$.

   Solution: As problem sheet 5, $\mathbf{r}(u,v) = \cos u\mathbf{i} + \sin u\mathbf{j} + v\mathbf{k}$ giving

$$\frac{\partial\mathbf{r}}{\partial u}\times\frac{\partial\mathbf{r}}{\partial v} = \cos u\mathbf{i} + \sin u\mathbf{j}. \tag{34}$$

   and

$$\mathbf{F}\cdot\frac{\partial\mathbf{r}}{\partial u}\times\frac{\partial\mathbf{r}}{\partial v} = (x+x^2)\cos u + y\sin u = \cos^2 + \cos^3 u + \sin^2 u = 1 + \cos^3 u. \tag{35}$$

   and so, noting that the $\mathbf{F}$ is perpendicular to the normal at both ends and so we need only include the curved surface

$$\int_S \mathbf{F}\cdot d\mathbf{A} = \int_0^1 dv \int_0^{2\pi} du\ (1+\cos^3 u) = 2\pi, \tag{36}$$

   since the $\cos^3 u$ integral is zero by symmetry.

   Note: This problem can be solved by noting that $x^2\mathbf{i}$ makes no contribution (by symmetry) and $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$ has flux 2π since $\mathbf{F}\cdot\mathbf{n} = 1$.

3. Find the flux of $\mathbf{F} = z^3\mathbf{k}$ upwards through the part of the sphere $x^2+y^2+z^2=a^2$ above the $z=0$ plane.

   Solution: To parametrize the sphere choose $(u,v) = (\theta,\phi)$, that is, spherical polar angles. Since the radius $r=a$ this gives $x(u,v) = a\sin u\cos v$, $y(u,v) = a\sin u\sin v$, $z(u,v) = a\cos u$ with $0 \le \theta \le \pi/2$ and $0 \le \phi < 2\pi$. Since $\mathbf{F}$ only has a non-zero z-component we just need

$$\begin{aligned} \left(\frac{\partial\mathbf{r}}{\partial u}\times\frac{\partial\mathbf{r}}{\partial v}\right)_3 &= \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} \\ &= a^2(\cos u\cos v\sin u\cos v + \cos u\sin v\sin u\sin v) \\ &= a^2\cos u\sin u, \end{aligned} \tag{37}$$

   which is positive. So the orientation is upwards. Now $F_z = z^3 = a^3\cos^3 u$, and so

$$\begin{aligned} \int_S \mathbf{F}\cdot d\mathbf{A} &= a^5\int_0^{\frac{1}{2}\pi} du \int_0^{2\pi} dv\ \cos^4 u\sin u \\ &= 2\pi a^5\int_0^{\frac{1}{2}\pi} du\ \cos^4 u\sin u \\ &= 2\pi a^5\cdot -\frac{\cos^5 u}{5}\Big]_0^{\frac{1}{2}\pi} = \frac{2\pi a^5}{5}. \end{aligned} \tag{38}$$

4. Consider the 'point vortex' vector field

$$\mathbf{F} = \frac{y}{x^2+y^2}\mathbf{i} - \frac{x}{x^2+y^2}\mathbf{j}.$$

   Show that curl $\mathbf{F} = 0$ away from the z-axis. Establish that $\mathbf{F}$ is *not* conservative in the (non simply-connected) domain $x^2+y^2 \ge \frac{1}{2}$. Is $\mathbf{F}$ conservative in the domain defined by $x^2+y^2 \ge \frac{1}{2}$, $y \ge 0$? If so obtain a scalar potential for $\mathbf{F}$.

   Solution:

$$\begin{aligned} \nabla\times\mathbf{F} &= \frac{1}{2}\mathbf{k}\left[\partial_x\left(\frac{-x}{x^2+y^2}\right) + \partial_y\left(\frac{y}{x^2+y^2}\right)\right] \\ &= \frac{1}{2}\mathbf{k}\left[-\frac{1}{x^2+y^2} + \frac{2x^2}{(x^2+y^2)^2} - \frac{1}{x^2+y^2} + \frac{2y^2}{(x^2+y^2)^2}\right] \\ &= 0. \end{aligned} \tag{39}$$

   To show that $\mathbf{F}$ is not conservative consider $\oint_C \mathbf{F}\cdot d\mathbf{l}$ where C is the unit circle. Using the obvious parametrization

$$\begin{aligned} \oint_C \mathbf{F}\cdot d\mathbf{l} &= \int_0^{2\pi} du\ \left(-\sin^2 u - \cos^2 u\right) \\ &= -2\pi \ne 0, \end{aligned} \tag{40}$$

therefore $\mathbf{F}$ is not conservative.

The domain $x^2+y^2 \geq \frac{1}{2}$, $y \geq 0$ is simply connected and $\mathbf{F}$ is irrotational and smooth is the domain. Thus $\mathbf{F}$ is conservative.

Write $\mathbf{F} = \nabla\phi$. Seek a $\phi(x,y)$ such that

$$\frac{\partial\phi}{\partial x} = \frac{y}{x^2+y^2}, \qquad \frac{\partial\phi}{\partial y} = -\frac{x}{x^2+y^2}. \tag{41}$$

Integrate first equation by treating y as a constant

$$\phi(x,y) = y\int \frac{dx}{x^2+y^2} = \tan^{-1}\frac{x}{y} + C(y). \tag{42}$$

Assume that x and y are non-negative, then

$$\tan^{-1}\frac{x}{y} + \tan^{-1}\frac{y}{x} = \frac{\pi}{2},$$

so that $\phi(x,y) = -\tan^{-1}\frac{y}{x}+$ a possibly y-dependent constant. However it is easy to check that $\phi = -\tan^{-1}\frac{y}{x}$ satisfies $\frac{\partial\phi}{\partial y} = -\frac{x}{x^2+y^2}$. Clearly, $\tan^{-1}\frac{y}{x}$ is the usual polar angle θ, that is $\phi = -\theta$.

Can try to extend this back to the original domain $x^2+y^2 \geq \frac{1}{2}$, but ϕ will suffer a branch cut discontinuity at, say $\theta = \frac{3}{2}\pi$.

5. Let D be a plane region with area A whose boundary is a piecewise smooth closed curve C. Use Green's theorem to prove that the centroid $(\bar{x}, \bar{y})$ of D is

$$\begin{aligned} \bar{x} &= \frac{1}{2A}\oint_C dy\, x^2 \\ \bar{y} &= -\frac{1}{2A}\oint_C dx\, y^2. \end{aligned} \tag{43}$$

Use this result to compute the centroid of a semi-circle (this was determined in the lectures using the more standard formula).

*Solution:*The centroid $(\bar{x}, \bar{y})$ of a plane region D is given by

$$\bar{x} = \frac{\int_D dA\, x}{A} \qquad \bar{y} = \frac{\int_D dA\, y}{A}. \tag{44}$$

If the boundary of D is a piecewise smooth closed curve C, Green's theorem reads

$$\int_D dA\left(\frac{\partial g(x,y)}{\partial x} - \frac{\partial f(x,y)}{\partial y}\right) = \oint_C \left(dx\, f(x,y) + dy\, g(x,y)\right), \tag{45}$$

where $f(x,y)$ and $g(x,y)$ are function with continuous first derivatives and the curve is oriented anti-clockwise. Now taking $g(x,y) = \frac{1}{2}x^2$ and $f(x,y) = 0$ yields

$$\int_D dA\, x = \frac{1}{2}\int_C dy\, x^2. \tag{46}$$

Inserting this into the formula for $\bar{x}$ gives

$$\bar{x} = \frac{1}{2A}\oint_C dy x^2 \tag{47}$$

Similarly the choice $f(x,y) = \frac{1}{2}y^2$, $g(x,y) = 0$ gives

$$\bar{y} = -\frac{1}{2A}\oint_C dx\, y^2 \tag{48}$$

Here C comprises the semi-circular arc plus the line segment joining $(-1,0)$ and $(1,0)$. The integral $\int_C dy\, x^2$ is zero since the positive x part of the arc integral cancels the negative x part. Also the integral along the line segment is zero. This implies that $\bar{x} = 0$.

For the other integral, $\int_C dx\, y^2$, only the arc contributes since $y = 0$ along the line segment. Now

$$\int_C dx\, y^2 = -\int_{C,\text{clockwise}} dx\, y^2 = -\int_{-1}^{1} dx\, (1-x^2) = -\frac{4}{3}. \tag{49}$$

Since $A = \frac{1}{2}\pi$ it follows that

$$\bar{y} = \frac{\frac{4}{3}}{2A} = \frac{4}{3\pi}. \tag{50}$$