

231 Outline Solutions Tutorial Sheet 1, 2 and 3.^{1,2}

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Problem Sheet 1

1. Rewrite the integral

$$I = \int_0^1 dx \int_1^{e^x} dy \phi(x, y) \quad (1)$$

as a double integral with the opposite order of integration.

Solution: The range of y values: $1 \leq y \leq e$. For a fixed y , x has the range $\log y \leq x \leq 1$. Hence

$$I = \int_1^e dy \int_{\log y}^1 dx \phi(x, y). \quad (2)$$

2. Consider the integral

$$I = \int_D dV \phi \quad (3)$$

where D is the interior of the ellipsoid defined by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1. \quad (4)$$

Write down I as an iterated triple integral.

Solution: Upper surface of ellipsoid is

$$z = +c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}. \quad (5)$$

whereas the lower surface is

$$z = -c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}} \quad (6)$$

The surfaces join at $z = 0$ where $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, this provides range of x and y integrations: $y = -b\sqrt{1 - \frac{x^2}{a^2}}$ to $y = +b\sqrt{1 - \frac{x^2}{a^2}}$ and $x = -a$ to $x = a$:

$$I = \int_{-a}^a dx \int_{-b\sqrt{1 - \frac{x^2}{a^2}}}^{+b\sqrt{1 - \frac{x^2}{a^2}}} dy \int_{-c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}}^{+c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}} dz \phi(x, y, z). \quad (7)$$

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²Including material from Chris Ford, to whom many thanks.

3. The Gaussian integral formula

$$\int_{-\infty}^{\infty} dx e^{-x^2} = \sqrt{\pi} \quad (8)$$

can be derived easily with the help of polar coordinates. The trick is to note that the *square* of the integral can be recast as a double integral over R^2 :

$$\left(\int_{-\infty}^{\infty} dx e^{-x^2} \right)^2 = \int_{R^2} dA e^{-x^2 - y^2}. \quad (9)$$

By changing to polar coordinates evaluate this integral.

Solution: After changing to polars and making sure to include the Jacobian $J = r$

$$\int_{R^2} dA e^{-x^2 - y^2} = \int_0^{2\pi} d\theta \int_0^{\infty} dr r e^{-r^2} \quad (10)$$

and then do this integral by substituting $u = r^2$ so $du = 2rdr$ to give

$$I^2 = \pi \int_0^{\infty} du e^{-u} = \pi \quad (11)$$

as required.

Problem Sheet 2

1. Compute the Jacobian of the transformation from cartesian to parabolic cylinder coordinates

$$x = \frac{1}{2}(u^2 - v^2), \quad y = uv. \quad (12)$$

Solution: Well

$$\begin{aligned} J &= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \\ &= \begin{vmatrix} u & -v \\ v & u \end{vmatrix} \\ &= u^2 + v^2. \end{aligned} \quad (13)$$

2. Determine the volume of the region enclosed by the cylinder $x^2 + y^2 = 4$ and the planes $y + z = 4$ and $z = 0$. *Suggestion: Use Cartesian coordinates.*

Solution: Range of integration: $z = 0$ to $z = 4 - y$, $y = -\sqrt{4 - x^2}$ to $y = +\sqrt{4 - x^2}$ and $x = -2$ to $x = 2$. Thus the volume is

$$V = \int_{-2}^2 dx \int_{-\sqrt{4 - x^2}}^{+\sqrt{4 - x^2}} dy \int_0^{4 - y} dz 1 = \int_{-2}^2 dx \int_{-\sqrt{4 - x^2}}^{+\sqrt{4 - x^2}} dy (4 - y), \quad (14)$$

the z integral being trivial. The y integral is also straightforward:

$$V = \int_{-2}^2 dx \, 8\sqrt{4-x^2} = 8 \cdot 2\pi = 16\pi. \quad (15)$$

The final integral can be evaluated by elementary means: either make the standard substitution ($x = 2\sin\theta$) or simply note that the integral represents the area of a semi-circle of radius 2.

3. Check that the Jacobian for the transformation from cartesian to spherical polar coordinates is

$$J = r^2 \sin\theta.$$

Consider the hemisphere defined by

$$\sqrt{x^2 + y^2 + z^2} \leq 1, \quad z \geq 0.$$

Using spherical polar coordinates compute its volume and centroid.

Solution: Spherical polar coordinates are defined by

$$\begin{aligned} x &= r \sin\theta \cos\phi, \\ y &= r \sin\theta \sin\phi, \\ z &= r \cos\theta. \end{aligned} \quad (16)$$

The Jacobian is

$$\begin{aligned} \frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} &= \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin\theta \cos\phi & r \cos\theta \cos\phi & -r \sin\theta \sin\phi \\ \sin\theta \sin\phi & r \cos\theta \sin\phi & r \sin\theta \cos\phi \\ \cos\theta & -r \sin\theta & 0 \end{vmatrix} \\ &= r^2 \sin^2\theta [\cos^2\theta \cos^2\phi + \cos^2\theta \sin^2\phi + \sin^2\theta \cos^2\phi + \sin^2\theta \sin^2\phi] \\ &= r^2 \sin\theta. \end{aligned} \quad (17)$$

Volume = $\int_D dV$. Centroid $\bar{x} = \bar{y} = 0$ by symmetry and $\bar{z} = \int_D dV \, z / \int_D dV$. Now

$$\begin{aligned} \int_D dV &= \int_0^{2\pi} d\phi \int_0^{\frac{1}{2}\pi} d\theta \int_0^1 dr \, r^2 \sin\theta = 2\pi \int_0^{\frac{1}{2}\pi} d\theta \sin\theta \frac{1}{3} \\ &= -\frac{2}{3} \pi \cos\theta \Big|_0^{\frac{1}{2}\pi} = 2\pi/3 \end{aligned} \quad (18)$$

as expected.

The other integral is

$$\begin{aligned} \int_D dV \, z &= \int_0^{2\pi} d\phi \int_0^{\frac{1}{2}\pi} d\theta \int_0^1 dr \, r^2 \sin\theta \cdot r \cos\theta = 2\pi \int_0^{\frac{1}{2}\pi} d\theta \sin\theta \cos\theta \frac{1}{4} \\ &= \frac{\pi}{2} \int_0^{\frac{1}{2}\pi} d\theta \frac{1}{2} \sin 2\theta = \frac{\pi}{4} \end{aligned} \quad (19)$$

and therefore $\bar{z} = 3/8$.

4. Determine the curl of the vector fields

$$(a) \mathbf{F} = -yz \sin x \mathbf{i} + z \cos x \mathbf{j} + y \cos x \mathbf{k}.$$

$$(b) \mathbf{F} = \frac{1}{2}y \mathbf{i} - \frac{1}{2}x \mathbf{j}.$$

Solution: The first one has $\text{curl } \mathbf{F} = y \sin x \mathbf{j} - z \sin x \mathbf{k}$ and the second $\text{curl } \mathbf{F} = -\mathbf{k}$.

5. Show that away from the origin the vector field

$$\mathbf{F} = \frac{\hat{\mathbf{r}}}{r^3} = \frac{\mathbf{r}}{r^3}$$

is divergenceless.

Solution: So

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x} \frac{x}{r^3} + \frac{\partial}{\partial y} \frac{y}{r^3} + \frac{\partial}{\partial z} \frac{z}{r^3} \quad (20)$$

Using the product rule

$$\frac{\partial}{\partial x} \frac{x}{r^3} = \frac{r^3 - 3x(x/r)r^2}{r^6} = \frac{1}{r^3} - \frac{3x^2}{r^5} \quad (21)$$

and so

$$\nabla \cdot \mathbf{F} = \frac{3}{r^3} - \frac{3(x^2 + y^2 + z^2)}{r^5} = 0 \quad (22)$$

using $r^2 = x^2 + y^2 + z^2$. Note by the way we have used

$$\frac{\partial}{\partial x} r = \frac{\partial}{\partial x} \sqrt{x^2 + y^2 + z^2} = \frac{x}{r} \quad (23)$$

using the chain rule.

Problem Sheet 3

1. Rewrite the integral

$$I = \int_0^1 dy \int_{\tan^{-1}y}^{\frac{\pi}{4}} dx \, \phi(x, y), \quad (24)$$

as an iterated double integral with the opposite order of integration. Compute the area of the region of integration.

Solution: Here $x = \frac{1}{4}\pi$ is the right boundary and $x = \tan^{-1}y$ is the left boundary. A quick sketch shows that the left boundary is also the upper boundary which can be written $y = \tan x$. The lower boundary is $y = 0$ and $0 \leq x \leq \frac{1}{4}\pi$. Thus

$$I = \int_0^{\frac{1}{4}\pi} dx \int_0^{\tan x} dy \, \phi(x, y). \quad (25)$$

Area obtained by setting $\phi(x, y) = 1$:

$$\begin{aligned}
 A &= \int_0^{\frac{1}{4}\pi} dx \int_0^{\tan x} dy \\
 &= \int_0^{\frac{1}{4}\pi} dx \tan x = -\log(\cos x) \Big|_0^{\frac{1}{4}\pi} \\
 &= -\left(\log \frac{1}{\sqrt{2}} - \log 1\right) \\
 &= \frac{\log 2}{2}.
 \end{aligned} \tag{26}$$

2. Compute the element of area for elliptic cylinder coordinates which are defined as

$$x = a \cosh u \cos v \tag{27}$$

$$y = a \sinh u \sin v. \tag{28}$$

Solution: $\delta A = J \delta u \delta v$ with

$$\begin{aligned}
 J &= \left\| \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \right\| = \left\| \begin{pmatrix} a \sinh u \cos v & -a \cosh u \sin v \\ a \cosh u \sin v & a \sinh u \cos v \end{pmatrix} \right\| \\
 &= a^2 (\sinh^2 u \cos^2 v + \cosh^2 u \sin^2 v)
 \end{aligned} \tag{29}$$

This can be simplified a bit:

$$\begin{aligned}
 J &= a^2 (\sinh^2 u \cos^2 v + \cosh^2 u \sin^2 v) = a^2 [\sinh^2 u (1 - \sin^2 v) + \cosh^2 u \sin^2 v] \\
 &= a^2 [\sinh^2 u + \sin^2 v (\cosh^2 u - \sinh^2 u)]
 \end{aligned} \tag{30}$$

Using $\cosh^2 u - \sinh^2 u = 1$ gives $J = a^2 (\sinh^2 u + \sin^2 v)$.

3. Compute the area and centroid of the plane region enclosed by the cardioid $r(\theta) = 1 + \cos \theta$ (r and θ are polar coordinates).

Solution: Use polar coordinates to evaluate area integral; θ ranges from 0 to 2π and r ranges from 0 to $1 + \cos \theta$ and the Jacobian is $J = r$

$$\begin{aligned}
 A &= \int_D dV = \int_0^{2\pi} d\theta \int_0^{1+\cos \theta} dr r \\
 &= \int_0^{2\pi} d\theta \frac{1}{2} (1 + \cos \theta)^2 \\
 &= \frac{1}{2} \int_0^{2\pi} d\theta (1 + 2 \cos \theta + \cos^2 \theta) \\
 &= \frac{1}{2} (2\pi + 0 + \pi) = \frac{3}{2}\pi,
 \end{aligned} \tag{31}$$

since $\cos \theta$ integrates to zero and the average value of $\cos^2 \theta$ is $\frac{1}{2}$.

Similarly

$$\begin{aligned}
 \int_D x dV &= \int_0^{2\pi} d\theta \int_0^{1+\cos \theta} dr r^2 \cos \theta \\
 &= \frac{1}{3} \int_0^{2\pi} d\theta \frac{1}{2} (1 + \cos \theta)^3 \cos \theta \\
 &= \frac{1}{3} \int_0^{2\pi} d\theta (\cos \theta + 3 \cos^2 \theta + 3 \cos^3 \theta + \cos^4 \theta) \\
 &= \frac{1}{3} \left(3\pi + 0 + \frac{3}{4}\pi \right) = \frac{5}{4}\pi,
 \end{aligned} \tag{32}$$

and so $\bar{x} = 5/6$. By symmetry $\bar{y} = 0$.

4. Show that away from the origin the vector field

$$\mathbf{F} = \frac{\hat{\mathbf{r}}}{r^2} = \frac{\mathbf{r}}{r^3} \tag{33}$$

is irrotational (here $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$).

Solution: Note that $\mathbf{F} = \text{grad}(-1/r)$ and so $\text{curl } \mathbf{F} = 0$. This can also be done by direct calculation.

5. Prove the identity

$$\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \Delta \mathbf{F}. \tag{34}$$

Solution: Lets do the first component:

$$[\nabla \times (\nabla \times \mathbf{F})]_1 = \frac{\partial}{\partial y} (F_{2,x} - F_{1,y}) - \frac{\partial}{\partial z} (F_{3,x} - F_{1,z}) \tag{35}$$

where $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ and I am using a comma notation for differentiation so for example

$$F_{2,x} = \frac{\partial F_2}{\partial x} \tag{36}$$

Now, taking away some brackets

$$[\nabla \times (\nabla \times \mathbf{F})]_1 = F_{2,xy} - F_{1,yy} - F_{3,xz} - F_{1,zz} \tag{37}$$

Coming from the other side

$$[\nabla(\nabla \cdot \mathbf{F})]_1 = \frac{\partial}{\partial x} (F_{1,x} + F_{2,y} + F_{3,z}) = F_{1,xx} + F_{2,yx} + F_{3,zx} \tag{38}$$

so

$$[\nabla \times (\nabla \times \mathbf{F})]_1 - [\nabla(\nabla \cdot \mathbf{F})]_1 = F_{1,xx} + F_{1,yy} + F_{1,zz} = [\Delta \mathbf{F}]_1 \tag{39}$$

and similarly for the other components.