231 Outline Solutions Tutorial Sheet 1, 2 and 3.[1][2]

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Problem Sheet 1

1. Rewrite the integral
$$I=\int_0^1 dx\int_1^{e^x} dy\ \phi(x,y) \tag{1}$$
as a double integral with the opposite order of integration.

   *Solution:*The range of y values: $1\le y\le e$. For a fixed y, x has the range $\log y\le x\le 1$. Hence
$$I=\int_1^e dy\int_{\log y}^1 dx\ \phi(x,y). \tag{2}$$

2. Consider the integral
$$I=\int_D dV\ \phi \tag{3}$$
where D is the interior of the ellipsoid defined by
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1. \tag{4}$$
Write down I as an iterated triple integral.

   *Solution:*Upper surface of ellipsoid ia
$$z=+c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}. \tag{5}$$
whereas the lower surface is
$$z=-c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}} \tag{6}$$
The surfaces join at $z=0$ where $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, this provides range of x and y integrations: $y=-b\sqrt{1-\frac{x^2}{a^2}}$ to $y=+b\sqrt{1-\frac{x^2}{a^2}}$ and $x=-a$ to $x=a$:
$$I=\int_{-a}^a dx\int_{-b\sqrt{1-\frac{x^2}{a^2}}}^{+b\sqrt{1-\frac{x^2}{a^2}}} dy\int_{-c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}^{+c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz\ \phi(x,y,z). \tag{7}$$

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[2]Including material from Chris Ford, to whom many thanks.

3. The Gaussian integral formula

$$\int_{-\infty}^{\infty} dx\ e^{-x^2} = \sqrt{\pi} \tag{8}$$

can be derived easily with the help of polar coordinates. The trick is to note that the *square* of the integral can be recast as a double integral over R^2:

$$\left(\int_{-\infty}^{\infty} dx\ e^{-x^2}\right)^2 = \int_{R^2} dA\ e^{-x^2-y^2}. \tag{9}$$

By changing to polar coordinates evaluate this integral.

*Solution:*After changing to polars and making sure to include the Jacobian $J = r$

$$\int_{R^2} dA\ e^{-x^2-y^2} = \int_0^{2\pi} d\theta \int_0^{\infty} dr\ re^{-r^2} \tag{10}$$

and then do this integral by substituting $u = r^2$ so $du = 2rdr$ to give

$$I^2 = \pi \int_0^{\infty} due^{-u} = \pi \tag{11}$$

as required.

Problem Sheet 2

1. Compute the Jacobian of the transformation from cartesian to parabolic cylinder coordinates

$$x = \frac{1}{2}(u^2 - v^2), \qquad y = uv. \tag{12}$$

*Solution:*Well

$$\begin{aligned} J &= \left|\left|\begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{matrix}\right|\right| \\ &= \left|\left|\begin{matrix} u & -v \\ v & u \end{matrix}\right|\right| \\ &= u^2 + v^2. \end{aligned} \tag{13}$$

2. Determine the volume of the region enclosed by the cylinder $x^2 + y^2 = 4$ and the planes $y + z = 4$ and $z = 0$. *Suggestion: Use Cartesian coordinates.*

*Solution:*Range of integration: $z = 0$ to $z = 4 - y$, $y = -\sqrt{4-x^2}$ to $y = +\sqrt{4-x^2}$ and $x = -2$ to $x = 2$. Thus the volume is

$$V = \int_{-2}^{2} dx \int_{-\sqrt{4-x^2}}^{+\sqrt{4-x^2}} dy \int_0^{4-y} dz\ 1 = \int_{-2}^{2} dx \int_{-\sqrt{4-x^2}}^{+\sqrt{4-x^2}} dy\ (4-y), \tag{14}$$

the z integral being trivial. The y integral is also straightforward:

$$V = \int_{-2}^{2} dx\ 8\sqrt{4-x^2} = 8 \cdot 2\pi = 16\pi. \tag{15}$$

The final integral can be evaluated by elementary means: either make the standard substitution ($x = 2\sin\theta$) or simply note that the integral represents the area of a semi-circle of radius 2.

3. Check that the Jacobian for the transformation from cartesian to spherical polar coordinates is

$$J = r^2 \sin\theta.$$

Consider the hemisphere defined by

$$\sqrt{x^2+y^2+z^2} \le 1, \qquad z \ge 0.$$

Using spherical polar coordinates compute its volume and centroid.

*Solution:*Spherical polar coordinates are defined by

$$\begin{aligned} x &= r\sin\theta\cos\phi, \\ y &= r\sin\theta\sin\phi, \\ z &= r\cos\theta. \end{aligned} \tag{16}$$

The Jacobian is

$$\begin{aligned} \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} &= \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}. \\ &= r^2\sin^2\theta\left[\cos^2\theta\cos^2\phi + \cos^2\theta^2\sin^2\phi^2 + \sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi\right] \\ &= r^2\sin\theta. \end{aligned} \tag{17}$$

Volume $= \int_D dV$. Centroid $\bar{x} = \bar{y} = 0$ by symmetry and $\bar{z} = \int_D dV\ z / \int_D dV$. Now

$$\begin{aligned} \int_D dV &= \int_0^{2\pi} d\phi \int_0^{\frac{1}{2\pi}} d\theta \int_0^1 dr\ r^2\sin\theta = 2\pi \int_0^{\frac{1}{2\pi}} d\theta\ \sin\theta\ \frac{1}{3}. \\ &= -\frac{2}{3}\pi\cos\theta\Big|_0^{\frac{1}{2\pi}} = 2\pi/3 \end{aligned} \tag{18}$$

as expected.

The other integral is

$$\begin{aligned} \int_D dV\ z &= \int_0^{2\pi} d\phi \int_0^{\frac{1}{2}\pi} d\theta \int_0^1 dr\ r^2\sin\theta \cdot r\cos\theta = 2\pi \int_0^{\frac{1}{2}\pi} d\theta\ \sin\theta\cos\theta\ \frac{1}{4} \\ &= \frac{\pi}{2}\int_0^{\frac{1}{2}\pi} d\theta\ \frac{1}{2}\sin 2\theta = \frac{\pi}{4} \end{aligned} \tag{19}$$

and therefore $\bar{z} = 3/8$.

4. Determine the curl of the vector fields

   (a) $\mathbf{F} = -yz\sin x\ \mathbf{i} + z\cos x\ \mathbf{j} + y\cos x\ \mathbf{k}$.

   (b) $\mathbf{F} = \frac{1}{2}y\ \mathbf{i} - \frac{1}{2}x\ \mathbf{j}$.

   *Solution:*The first one has curl $\mathbf{F} = y\sin x\mathbf{j} - z\sin x\mathbf{k}$ and the second curl $\mathbf{F} = -\mathbf{k}$.

5. Show that away from the origin the vector field

$$\mathbf{F} = \frac{\hat{\mathbf{r}}}{r^3} = \frac{\mathbf{r}}{r^3}$$

   is divergenceless.

   *Solution:*So

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}\frac{x}{r^3} + \frac{\partial}{\partial y}\frac{y}{r^3} + \frac{\partial}{\partial y}\frac{y}{r^3} \tag{20}$$

   Using the product rule

$$\frac{\partial}{\partial x}\frac{x}{r^3} = \frac{r^3 - 3x(x/r)r^2}{r^6} = \frac{1}{r^3} - \frac{3x^2}{r^5} \tag{21}$$

   and so

$$\nabla \cdot \mathbf{F} = \frac{3}{r^3} - \frac{3(x^2+y^2+z^2)}{r^5} = 0 \tag{22}$$

   using $r^2 = x^2 + y^2 + z^2$. Note by the way we have used

$$\frac{\partial}{\partial x}r = \frac{\partial}{\partial x}\sqrt{x^2+y^2+z^2} = \frac{x}{r} \tag{23}$$

   using the chain rule.

Problem Sheet 3

1. Rewrite the integral

$$I = \int_0^1 dy \int_{\tan^{-1} y}^{\frac{\pi}{4}} dx\ \phi(x,y), \tag{24}$$

   as an iterated double integral with the opposite order of integration. Compute the area of the region of integration.

   *Solution:*Here $x = \frac{1}{4}\pi$ is the right boundary and $x = \tan^{-1} y$ is the left boundary. A quick sketch shows that the left boundary is also the upper boundary which can be written $y = \tan x$. The lower boundary is $y = 0$ and $0 \leq x \leq \frac{1}{4}\pi$.Thus

$$I = \int_0^{\frac{1}{4}\pi} dx \int_0^{\tan x} dy\ \phi(x,y). \tag{25}$$

Area obtained by setting $\phi(x,y)=1$:

$$\begin{aligned} A &= \int_0^{\frac{1}{4}\pi} dx \int_0^{\tan x} dy \\ &= \int_0^{\frac{1}{4}\pi} dx\ \tan x = -\log(\cos x)|_0^{\frac{1}{4}\pi} \\ &= -\left(\log\frac{1}{\sqrt{2}} - \log 1\right) \\ &= \frac{\log 2}{2}. \end{aligned} \tag{26}$$

2. Compute the element of area for elliptic cylinder coordinates which are defined as

$$x = a\cosh u\cos v \tag{27}$$
$$y = a\sinh u\sin v. \tag{28}$$

Solution: $\delta A = J\delta u\delta v$ with

$$\begin{aligned} J &= \left\| \begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{matrix} \right\| = \left\| \begin{matrix} a\sinh u\cos v & -a\cosh u\sin v \\ a\cosh u\sin v & a\sinh u\cos v \end{matrix} \right\| \\ &= a^2(\sinh^2 u\cos^2 v + \cosh^2 u\sin^2 v) \end{aligned} \tag{29}$$

This can be simplified a bit:

$$\begin{aligned} J &= a^2(\sinh^2 u\cos^2 v + \cosh^2 u\sin^2 v) = a^2[\sinh^2 u(1-\sin^2 v) + \cosh^2 u\sin^2 v] \\ &= a^2[\sinh^2 + \sin^2 v(\cosh^2 - \sinh^2 u)] \end{aligned} \tag{30}$$

Using $\cosh^2 u - \sinh^2 u = 1$ gives $J = a^2(\sinh^2 u + \sin^2 v)$.

3. Compute the area and centroid of the plane region enclosed by the cardioid $r(\theta) = 1+\cos\theta$ (r and θ are polar coordinates).

   Solution: Use polar coördinates to evaluate area integral; θ ranges from 0 to 2π and r ranges from 0 to $1+\cos\theta$ and the Jacobian is $J=r$

$$\begin{aligned} A &= \int_D dV = \int_0^{2\pi} d\theta \int_0^{1+\cos\theta} dr\ r \\ &= \int_0^{2\pi} d\theta\ \frac{1}{2}(1+\cos\theta)^2 \\ &= \frac{1}{2}\int_0^{2\pi} d\theta\ (1+2\cos\theta+\cos^2\theta) \\ &= \frac{1}{2}(2\pi+0+\pi) = \frac{3}{2}\pi, \end{aligned} \tag{31}$$

since $\cos\theta$ integrates to zero and the average value of $\cos^2\theta$ is $\frac{1}{2}$.

Similarily

$$
\begin{aligned}
\int_D x dV &= \int_0^{2\pi} d\theta \int_0^{1+\cos\theta} dr\ r^2\cos\theta \\
&= \frac{1}{3}\int_0^{2\pi} d\theta\ \frac{1}{2}(1+\cos\theta)^3\cos\theta \\
&= \frac{1}{3}\int_0^{2\pi} d\theta\ (\cos\theta + 3\cos^2\theta + 3\cos^3\theta + \cos^4\theta) \\
&= \frac{1}{3}\left(3\pi + 0 + \frac{3}{4}\pi\right) = \frac{5}{4}\pi, \qquad (32)
\end{aligned}
$$

and so $\bar{x} = 5/6$. By symmetry $\bar{y} = 0$.

4. Show that away from the origin the vector field

$$\mathbf{F} = \frac{\hat{\mathbf{r}}}{r^2} = \frac{\mathbf{r}}{r^3} \qquad (33)$$

is irrotational (here $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $r = |\mathbf{r}| = \sqrt{x^2+y^2+z^2}$).

*Solution:*Note that $\mathbf{F} = \text{grad}\ (-1/r)$ and so curl $\mathbf{F} = 0$. This can also be done by direct calculation.

5. Prove the identity

$$\nabla\times(\nabla\times\mathbf{F}) = \nabla(\nabla\cdot\mathbf{F}) - \triangle\mathbf{F}. \qquad (34)$$

*Solution:*Lets do the first component:

$$[\nabla\times(\nabla\times\mathbf{F})]_1 = \frac{\partial}{\partial y}(F_{2,x} - F_{1,y}) - \frac{\partial}{\partial z}(F_{3,x} - F_{1,z}) \qquad (35)$$

where $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ and I am using a comma notation for differenciation so for example

$$F_{2,x} = \frac{\partial F_2}{\partial x} \qquad (36)$$

Now, taking away some brackets

$$[\nabla\times(\nabla\times\mathbf{F})]_1 = F_{2,xy} - F_{1,yy} - F_{3,xz} - F_{1,zz} \qquad (37)$$

Coming from the other side

$$[\nabla(\nabla\cdot\mathbf{F})]_1 = \frac{\partial}{\partial x}(F_{1,x} + F_{2,y} + F_{3,z}) = F_{1,xx} + F_{2,yx} + F_{3,zx} \qquad (38)$$

so

$$[\nabla\times(\nabla\times\mathbf{F})]_1 - [\nabla(\nabla\cdot\mathbf{F})]_1 = F_{1,xx} + F_{1,yy} + F_{1,zz} = [\triangle\mathbf{F}]_1 \qquad (39)$$

and similarily for the other components.