231 Tutorial Sheet 9: due Friday January 13[1,2]

9 December 2005

Useful facts:

- The Gauss Divergence Theorem: If D is connected region in $\mathbf{R}^3$ with a piecewise smooth surface S oriented to point out of D and if $\mathbf{F}$ is a vector field defined in a region containing D and with continuous derivatives then

$$\int_D dV \nabla \cdot \mathbf{F} = \int_S \mathbf{F} \cdot \mathbf{dA} \tag{1}$$

- The vector potential: on a domain with no internal boundary, a smooth vector vector field $\mathbf{F}$ with zero divergence has a vector field $\mathbf{A}$ so that $\mathbf{F} = \nabla \times \mathbf{A}$.

- Line and surface integrals of a scalar:

$$\begin{aligned} \int_C dl\phi &= \int_C |\mathbf{dl}|\phi \\ \int_S dA\phi &= \int_S |\mathbf{dA}|\phi \end{aligned} \tag{2}$$

and, in parameterized form, with the obvious notation

$$\begin{aligned} \mathbf{dl} &= \frac{d\mathbf{r}}{dt}dt \\ \mathbf{dA} &= \frac{d\mathbf{r}}{du} \times \frac{d\mathbf{r}}{dv}du\,dv \end{aligned} \tag{3}$$

[1] Conor Houghton, `houghton@maths.tcd.ie`, see also `http://www.maths.tcd.ie/~houghton/231`

[2] Including material from Chris Ford, to whom many thanks.

Questions

1. Using Gauss' theorem or otherwise compute the flux of the vector field $\mathbf{F} = x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ upwards through the part of the hyperboloid, $z^2 - x^2 - y^2 = 1$, lying between the $z = 1$ and $z = 2$ planes. Note that the surface to be integrated over is not a closed surface. Thus to apply Gauss' theorem consider the closed surface comprising the given surface and a suitable disc in the $z = 2$ plane. Using Gauss' theorem the flux out of the combined closed surface can be calculated. Then the disc contribution should be subtracted.

2. Obtain vector potentials for the solenoidal vector fields:

   (a) $\mathbf{F} = \mathbf{i} + 2z\mathbf{j} + \mathbf{k}$

   (b) $\mathbf{F} = e^z\mathbf{i} + e^x\mathbf{k}$.

3. Compute the lengths of the following curves

   (a) A helix with parametrization $x(u) = 2\cos u$, $y(u) = 2\sin u$, $z(u) = u$ and $0 \le u \le 2\pi$.

   (b) A lemniscate $r^2 = \cos 2\theta$ with $-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$ (r and θ are polar coordinates).

   In part ii) write the result as an integral (don't bother trying to compute it).

4. Determine the surface area of the 'bowl' which is the part of the paraboloid $z = x^2+y^2$ below the $z = 1$ plane.

5. The centroid of a curve C, $(\bar{x}, \bar{y}, \bar{z})$ is given by

$$\bar{x} = \frac{1}{L}\int_C x dl \tag{4}$$

   and so on, with L the length of the curve. Show that the centroid of a circular arc with radius r and angle 2θ (at the centre) is at a distance of $r\sin\theta/\theta$ from the centre.