231 Tutorial Sheet 7[1][2]

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Useful facts:

- The Gauss Divergence Theorem: If D is connected region in $\mathbf{R}^3$ with a piecewise smooth surface S oriented to point out of D and if $\mathbf{F}$ is a vector field defined in a region containing D and with continuous derivatives then

$$\int_D dV \nabla \cdot \mathbf{F} = \int_S \mathbf{F} \cdot \mathbf{dA} \tag{1}$$

Questions

1. Which of the following vector fields are conservative?

   (a) $\mathbf{F} = -yz \sin x \ \mathbf{i} + z \cos x \ \mathbf{j} + y \cos x \ \mathbf{k}$.

   (b) $\mathbf{F} = \frac{1}{2} y \ \mathbf{i} - \frac{1}{2} x \ \mathbf{j}$.

   (c) $\mathbf{F} = \frac{1}{2}(\mathbf{B} \times \mathbf{r})$ where $\mathbf{B}$ is a constant vector.

2. Using Gauss' theorem or otherwise compute the flux of the vector field $\mathbf{F} = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$ through the hemisphere $x^2 + y^2 + z^2 = 1$, $z \geq 0$ with the orientation taken upwards. What is the flux out of the whole sphere?

3. Consider, again, the vector field

$$\mathbf{F} = \frac{\mathbf{r}}{r^3}, \qquad \mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

   (a) Compute the flux of $\mathbf{F}$ out of a sphere of radius a centred at the origin.

   (b) Compute the flux of $\mathbf{F}$ out of the box $1 \leq x \leq 2$, $0 \leq y \leq 1$, $0 \leq z \leq 1$.

   (c) Compute the flux of $\mathbf{F}$ out of the box $-1 \leq x \leq 1$, $-1 \leq y \leq 1$, $-1 \leq z \leq 1$.

[1]Conor Houghton, houghton@maths.tcd.ie, see also http://www.maths.tcd.ie/~houghton/231

[2]Including material from Chris Ford, to whom many thanks.