231 Tutorial Sheet 6: due Friday November 25[1,2]

18 November 2005

Useful facts:

- To evaluate the line integral for a parameterized curve:

$$\int_c \mathbf{F}\cdot\mathbf{dl} = \int_{t_1}^{t_2} \mathbf{F}\cdot\frac{d\mathbf{r}}{dt}dt \tag{1}$$

  where t_1 and t_2 are the parameter values corresponding to the begining and end of the curve.

- To evaluate the surface integral for a parameterized surface:

$$\int\int_S \mathbf{F}\cdot\mathbf{dA} = \int\int_S \mathbf{F}\cdot\left(\frac{d\mathbf{r}}{du}\times\frac{d\mathbf{r}}{dv}\right)dudv \tag{2}$$

- Stokes' theorem: for an orientable piecewise smooth surface S with an orientable piecewise smooth boundy C oriented so that $\mathbf{N}\times\mathbf{dl}$ points into S with $\mathbf{n}$ the normal and $\mathbf{dl}$ a tangent to C and $\mathbf{F}$ a vector field defined in a region containing S, then

$$\int_S \text{curl}\,\mathbf{F}\cdot\mathbf{dA} = \int_C \mathbf{F}\cdot\mathbf{dl} \tag{3}$$

- If S is a surface δS is its bounding curve with the orientation describe above.

- Green's theorem on the plane: led D be a region in the xy-plane bounded by a piecewise continuous curve C, if $f(x,y)$ and $g(x,y)$ have continuous first derivatives

$$\int_D \left(\frac{\partial g}{\partial x}-\frac{\partial f}{\partial y}\right)dxdy = \int_C (fdx+gdy) \tag{4}$$

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[2]Including material from Chris Ford, to whom many thanks.

Questions

1. For each of the following vector fields compute the line integral $\oint_C \mathbf{F} \cdot \mathbf{dl}$ where C is the unit circle in the $x - y$ plane taken anti-clockwise.

   (a) $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$

   (b) $\mathbf{F} = y\mathbf{i} - x^2 y\mathbf{j}$.

2. Compute the flux of the vector field $\mathbf{F} = (x + x^2)\mathbf{i} + y\mathbf{j}$ out of the cylinder defined by $x^2 + y^2 = 1$ and $0 \leq z \leq 1$.

3. Find the flux of $\mathbf{F} = z^3\mathbf{k}$ upwards through the part of the sphere $x^2 + y^2 + z^2 = a^2$ above the $z = 0$ plane.

4. Consider the 'point vortex' vector field

$$\mathbf{F} = \frac{y}{x^2 + y^2}\mathbf{i} - \frac{x}{x^2 + y^2}\mathbf{j}.$$

   Show that curl $\mathbf{F} = 0$ away from the z-axis. Establish that $\mathbf{F}$ is *not* conservative in the (non simply-connected) domain $x^2 + y^2 \geq \frac{1}{2}$. Is $\mathbf{F}$ conservative in the domain defined by $x^2 + y^2 \geq \frac{1}{2}$, $y \geq 0$? If so obtain a scalar potential for $\mathbf{F}$.

5. Let D be a plane region with area A whose boundary is a piecewise smooth closed curve C. Use Green's theorem to prove that the centroid $(\bar{x}, \bar{y})$ of D is

$$\begin{aligned} \bar{x} &= \frac{1}{2A} \oint_C dy\, x^2 \\ \bar{y} &= -\frac{1}{2A} \oint_C dx\, y^2. \end{aligned} \tag{5}$$

   Use this result to compute the centroid of a semi-circle (this was determined in the lectures using the more standard formula).