231 Tutorial Sheet 4[1,2]

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Useful facts:

- For a vector field $\mathbf{F} = (F_1, F_2, F_3)$ the curl is

$$\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \tag{1}$$

- To evaluate the line integral for a parameterized curve:

$$\int_c \mathbf{F}\cdot\mathbf{dl} = \int_{t_1}^{t_2}\mathbf{F}\cdot\frac{d\mathbf{r}}{dt}dt \tag{2}$$

  where t_1 and t_2 are the parameter values corresponding to the begining and end of the curve.

- A vector field is *solinoidal* if it has zero divergence.
- A vector field is *irrotational* if it has zero curl.
- A vector field $\mathbf{F}$ is *conservative* if there is a scalar field ϕ such that

$$\mathbf{F} = \nabla\phi \tag{3}$$

[1] Conor Houghton, `houghton@maths.tcd.ie`, see also `http://www.maths.tcd.ie/~houghton/231`

[2] Including material from Chris Ford, to whom many thanks.

Questions

1. Compute the line integrals:
   (a) $\int_C (dx\ xy\ + \frac{1}{2}dy\ x^2\ + dz)$ where C is the line segment joining the origin and the point $(1, 1, 2)$.
   (b) $\int_C (dx\ yz\ + dy\ xz\ + dz\ yx^2)$ where C is the same line as in the previous part
2. Is $F = \mathbf{r}/r$ irrotational? Is it conservative. Is it conservative on a restricted domain? It is path independent?
3. Consider the vector field

$$\mathbf{F} = \frac{\frac{1}{2}y}{x^2+y^2}\mathbf{i} - \frac{\frac{1}{2}x}{x^2+y^2}\mathbf{j}. \tag{4}$$

   (a) Determine the line integral $\oint_C \mathbf{dl}\cdot\mathbf{F}$ where C is the unit circle centred at the origin in the $z = 0$ plane (taken anti-clockwise).
   (b) Compute the curl of $\mathbf{F}$.
   (c) Is $\mathbf{F}$ a conservative vector field?