231 Tutorial Sheet 3: due Friday November 4[1,2]

21 October 2005

Useful facts:

- Some trigonometric integrals are required. In particular you may quote the integrals:

$$\begin{aligned}\int_0^{2\pi} d\theta \ \cos\theta &= 0,\\ \int_0^{2\pi} d\theta \ \cos^2\theta &= \pi,\\ \int_0^{2\pi} d\theta \ \cos^3\theta &= 0\\ \int_0^{2\pi} d\theta \ \cos^4\theta &= \frac{3}{4}\pi \end{aligned} \tag{1}$$

  Two are zero by symmetry, the other two can be computed through standard trigonometric identities or via complex exponentials:

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}). \tag{2}$$

- For a scalar field ϕ the gradiant is

$$\text{grad}\,\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k} \tag{3}$$

- For a vector field $\mathbf{F} = (F_1, F_2, F_3)$ the divergence is

$$\text{div}\,\mathbf{F} = \nabla\cdot\mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \tag{4}$$

- For a vector field $\mathbf{F} = (F_1, F_2, F_3)$ the curl is

$$\text{curl}\,\mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \tag{5}$$

- The Laplacian:

$$\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{6}$$

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[2] Including material from Chris Ford, to whom many thanks.

Questions

1. Rewrite the integral
$$I = \int_0^1 dy \int_{\tan^{-1} y}^{\frac{\pi}{4}} dx\ \phi(x,y), \tag{7}$$
as an iterated double integral with the opposite order of integration. Compute the area of the region of integration.

2. Compute the element of area for elliptic cylinder coordinates which are defined as
$$x = a \cosh u \cos v \tag{8}$$
$$y = a \sinh u \sin v. \tag{9}$$

3. Compute the area and centroid of the plane region enclosed by the cardioid $r(\theta) = 1 + \cos\theta$ (r and θ are polar coordinates).

4. Show that away from the origin the vector field
$$\mathbf{F} = \frac{\hat{\mathbf{r}}}{r^2} = \frac{\mathbf{r}}{r^3} \tag{10}$$
is irrotational (here $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$).

5. Prove the identity
$$\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \triangle \mathbf{F}. \tag{11}$$