231 Tutorial Sheet 2[1,2]

21 October 2005

Useful facts:

- The Jacobian in three-dimensions:

$$dx_1 dx_2 d_3 = J dy_1 dy_2 dy_3 \tag{1}$$

where

$$J = \frac{\partial(x_1, x_2, x_3)}{\partial(y_1, y_2, y_3)} = \left\| \begin{matrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \frac{\partial x_1}{\partial y_3} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \frac{\partial x_2}{\partial y_3} \\ \frac{\partial x_3}{\partial y_1} & \frac{\partial x_3}{\partial y_2} & \frac{\partial x_3}{\partial y_3} \end{matrix} \right\| \tag{2}$$

- For a scalar field ϕ the gradiant is

$$\operatorname{grad} \phi = \frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k} \tag{3}$$

- If $\mathbf{a}$ is a vector $\hat{\mathbf{a}}$ is the corresponding unit vector

$$\hat{\mathbf{a}} = \frac{1}{|\mathbf{a}|}\mathbf{a} \tag{4}$$

- The direction derivative of a scalar field f in the $\mathbf{a}$ direction is $D_{\mathbf{a}} f = \hat{\mathbf{a}} \cdot \nabla f$.
- For a vector field $\mathbf{F} = (F_1, F_2, F_3)$ the divergence is

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \tag{5}$$

- For a vector field $\mathbf{F} = (F_1, F_2, F_3)$ the curl is

$$\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \tag{6}$$

[1] Conor Houghton, houghton@maths.tcd.ie, see also http://www.maths.tcd.ie/~houghton/231
[2] Including material from Chris Ford, to whom many thanks.

Questions

1. Compute the Jacobian of the transformation from cartesian to parabolic cylinder coordinates
$$x = \frac{1}{2}(u^2 - v^2), \quad y = uv.$$
2. 2. Determine the volume of the region enclosed by the cylinder $x^2 + y^2 = 4$ and the planes $y + z = 4$ and $z = 0$. *Suggestion: Use Cartesian coordinates.*
3. Check that the Jacobian for the transformation from cartesian to spherical polar coordinates is
$$J = r^2 \sin\theta.$$
Consider the hemisphere defined by
$$\sqrt{x^2 + y^2 + z^2} \le 1, \qquad z \ge 0.$$
Using spherical polar coordinates compute its volume and centroid.
4. Determine the curl of the vector fields
   (a) $\mathbf{F} = -yz \sin x\ \mathbf{i} + z \cos x\ \mathbf{j} + y \cos x\ \mathbf{k}$.
   (b) $\mathbf{F} = \frac{1}{2} y\ \mathbf{i} - \frac{1}{2} x\ \mathbf{j}$.
5. Show that away from the origin the vector field
$$\mathbf{F} = \frac{\hat{\mathbf{r}}}{r^3} = \frac{\mathbf{r}}{r^3}$$
is divergenceless.