**231** Tutorial Sheet  $2^{12}$ 

## 21 October 2005

## Useful facts:

• The Jacobian in three-dimensions:

$$dx_1 dx_2 d_3 = J dy_1 dy_2 dy_3 \tag{1}$$

where

$$J = \frac{\partial(x_1, x_2, x_3)}{\partial(y_1, y_2, y_3)} = \begin{vmatrix} \frac{\overline{\partial y_1}}{\partial y_1} & \overline{\partial y_2} & \overline{\partial y_3} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \frac{\partial x_2}{\partial y_3} \\ \frac{\partial x_3}{\partial y_1} & \frac{\partial x_3}{\partial y_2} & \frac{\partial x_3}{\partial y_3} \end{vmatrix}$$
(2)

 $|| \partial x_1 \quad \partial x_1 \quad \partial x_1 \quad ||$ 

• For a scalar field  $\phi$  the gradiant is

$$\operatorname{grad} \phi = \frac{\partial \phi}{\partial x} \mathbf{i} + \frac{\partial \phi}{\partial y} \mathbf{j} + \frac{\partial \phi}{\partial z} \mathbf{k}$$
(3)

• If  $\mathbf{a}$  is a vector  $\hat{\mathbf{a}}$  is the corresponding unit vector

$$\hat{\mathbf{a}} = \frac{1}{|\mathbf{a}|} \mathbf{a} \tag{4}$$

- The direction derivative of a scalar field f in the **a** direction is  $D_{\mathbf{a}}f = \hat{\mathbf{a}} \cdot \nabla f$ .
- For a vector field  $\mathbf{F} = (F_1, F_2, F_3)$  the divergence is

div 
$$\mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$
 (5)

• For a vector field  $\mathbf{F} = (F_1, F_2, F_3)$  the curl is

<sup>2</sup>Including material from Chris Ford, to whom many thanks.

$$\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$
(6)

- Questions
  - 1. Compute the Jacobian of the transformation from cartesian to parabolic cylinder coordinates

$$x = \frac{1}{2}(u^2 - v^2), \quad y = uv$$

- 2. 2. Determine the volume of the region enclosed by the cylinder  $x^2 + y^2 = 4$  and the planes y + z = 4 and z = 0. Suggestion: Use Cartesian coordinates.
- 3. Check that the Jacobian for the transformation from cartesian to spherical polar coordinates is

$$J = r^2 \sin \theta$$

Consider the hemisphere defined by

$$\sqrt{x^2 + y^2 + z^2} \le 1, \qquad z \ge 0.$$

Using spherical polar coordinates compute its volume and centroid.

4. Determine the curl of the vector fields

(a) 
$$\mathbf{F} = -yz \sin x \mathbf{i} + z \cos x \mathbf{j} + y \cos x \mathbf{k}.$$
  
(b)  $\mathbf{F} = \frac{1}{2}y \mathbf{i} - \frac{1}{2}x \mathbf{j}.$ 

5. Show that away from the origin the vector field

$$\mathbf{F} = \frac{\hat{\mathbf{r}}}{r^3} = \frac{\mathbf{r}}{r^3}$$

is divergenceless.

<sup>1</sup>Conor Houghton, houghton@maths.tcd.ie, see also http://www.maths.tcd.ie/~houghton/231