231 Tutorial Sheet 14.¹²

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Useful facts:

• The Gauss Divergence Theorem: If D is connected region in \mathbb{R}^3 with a piecewise smooth surface S oriented to point out of D and if \mathbf{F} is a vector field defined in a region containing D and with continuous derivatives then

$$\int_{D} dV \nabla \cdot \mathbf{F} = \int_{S} \mathbf{F} \cdot \mathbf{dA}$$
(1)

• To solve the equation y' + py = f multiply across by an integrating factor $\exp\left(\int^t d\tau p(\tau)\right)$ and express the right hand side as the derivative of a product.

Questions

1. In the lectures (quite a while ago) it was shown that the scalar field

$$\phi(\mathbf{r}) = \frac{1}{r},$$

where $r = \sqrt{x^2 + y^2 + z^2}$ is harmonic except at the origin. In fact it can be shown that

$$\nabla^2 \phi(\mathbf{r}) = -4\pi \delta^3(\mathbf{r}). \tag{A}$$

Formally apply Gauss' theorem to the vector field $\mathbf{F} = \nabla \phi$ to show that

$$\int_{r$$

This is clearly consistent with (A). Another treatment would replace the singular scalar field ϕ with a sequence of smooth scalar fields, e.g.

$$\phi_n(\mathbf{r}) = \frac{n}{\sqrt{n^2 r^2 + 1}}.$$

Prove that

$$\int_{R^3} dV \,\nabla^2 \phi_n(\mathbf{r}) = -4\pi.$$

Suggestion: Use Gauss' theorem to perform the integral for r < a instead of R^3 . Then take the $a \to \infty$ limit.

2. Solve y' - y = x.

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