

## 231 Tutorial Sheet 13.<sup>12</sup>

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### Useful facts:

- The Fourier integral or Fourier transform:

$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty} dk \widetilde{f(k)} e^{ikx} \\ \widetilde{f(k)} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dx f(x) e^{-ikx} \end{aligned}$$

- The Dirac delta function:

$$\int_{-\infty}^{\infty} dx f(x) \delta(x) = f(0)$$

- If  $h(x)$  is a continuous function

$$\int_{-\infty}^{\infty} dx f(x) \delta[h(x)] = \sum_{x_i: f(x_i)=0} \frac{f(x_i)}{|h'(x_i)|}$$

### Questions

- Express the following functions as Fourier integrals:

(a)

$$f(x) = \begin{cases} \cos x & |x| < \frac{\pi}{2} \\ 0 & |x| > \frac{\pi}{2} \end{cases}$$

(b)

$$f(x) = \frac{\sin x}{x}$$

- Prove the following properties of the Fourier transform

- The Fourier transform of an even function is even.
- The Fourier transform of a real odd function is purely imaginary.
- $\widehat{f'}(k) = ik\widehat{f}(k)$ .
- Acting with the Fourier transform four times reproduces the original function apart from an overall constant.

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<sup>2</sup>Including material from Chris Ford, to whom many thanks.

- Compute

(a)

$$\int_{-\infty}^{\infty} dx x^2 \delta(x-3)$$

(b)

$$\int_{-\infty}^{\infty} dx \delta(x^2+x)$$

(c)

$$\int_0^2 dx e^x \delta'(x-1)$$

(d)

$$\int_0^{\infty} dx e^{-ax} \delta(\cos x)$$

(e)

$$\int_0^{\infty} dx \delta(e^{ax} \cos x).$$

(f)

$$\frac{d}{dx} e^{a\theta(x)}.$$

where  $a$  is a constant.