231 Tutorial Sheet 13.[1,2]

3 Febuary 2006

Useful facts:

- The Fourier integral or Fourier transform:

$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty} dk\, \widetilde{f(k)} e^{ikx} \\ \widetilde{f(k)} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dx\, f(x) e^{-ikx} \end{aligned}$$

- The Dirac delta function:

$$\int_{-\infty}^{\infty} dx\, f(x)\delta(x) = f(0)$$

- If $h(x)$ is a continuous function

$$\int_{-\infty}^{\infty} dx\, f(x)\delta[h(x)] = \sum_{x_i : f(x_i)=0} \frac{f(x_i)}{|h'(x_i)|}$$

Questions

1. Express the following functions as Fourier integrals:

   (a)
   $$f(x) = \begin{cases} \cos x & |x| < \frac{\pi}{2} \\ 0 & |x| > \frac{\pi}{2} \end{cases}$$

   (b)
   $$f(x) = \frac{\sin x}{x}$$

2. Prove the following properties of the Fourier transform

   (a) The Fourier transform of an even function is even.

   (b) The Fourier transform of a real odd function is purely

   (c) $\tilde{f}'(k) = ik\tilde{f}(k)$.

   (d) Acting with the Fourier transform four times reproduces the original function apart from an overall constant.

[1]Conor Houghton, `houghton@maths.tcd.ie`, see also `http://www.maths.tcd.ie/~houghton/231`
[2]Including material from Chris Ford, to whom many thanks.

3. Compute

(a)

$$\int_{-\infty}^{\infty} dx\ x^2\ \delta(x-3)$$

(b)

$$\int_{-\infty}^{\infty} dx\ \delta(x^2+x)$$

(c)

$$\int_{0}^{2} dx\ e^x\ \delta'(x-1)$$

(d)

$$\int_{0}^{\infty} dx\ e^{-ax}\delta(\cos x)$$

(e)

$$\int_{0}^{\infty} dx\ \delta(e^{ax}\cos x).$$

(f)

$$\frac{d}{dx}e^{a\theta(x)}.$$

where a is a constant.