

231 Tutorial Sheet 13.¹²

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Useful facts:

- The Fourier integral or Fourier transform:

$$\begin{aligned}f(x) &= \int_{-\infty}^{\infty} dk \widetilde{f(k)} e^{ikx} \\ \widetilde{f(k)} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dx f(x) e^{-ikx}\end{aligned}$$

- The Dirac delta function:

$$\int_{-\infty}^{\infty} dx f(x) \delta(x) = f(0)$$

- If $h(x)$ is a continuous function

$$\int_{-\infty}^{\infty} dx f(x) \delta[h(x)] = \sum_{x_i: f(x_i)=0} \frac{f(x_i)}{|h'(x_i)|}$$

Questions

1. Express the following functions as Fourier integrals:

(a)

$$f(x) = \begin{cases} \cos x & |x| < \frac{\pi}{2} \\ 0 & |x| > \frac{\pi}{2} \end{cases}$$

(b)

$$f(x) = \frac{\sin x}{x}$$

2. Prove the following properties of the Fourier transform

(a) The Fourier transform of an even function is even.

(b) The Fourier transform of a real odd function is purely

(c) $\tilde{f}'(k) = ik\tilde{f}(k)$.

(d) Acting with the Fourier transform four times reproduces the original function apart from an overall constant.

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²Including material from Chris Ford, to whom many thanks.

3. Compute

(a)

$$\int_{-\infty}^{\infty} dx \, x^2 \, \delta(x-3)$$

(b)

$$\int_{-\infty}^{\infty} dx \, \delta(x^2+x)$$

(c)

$$\int_0^2 dx \, e^x \, \delta'(x-1)$$

(d)

$$\int_0^{\infty} dx \, e^{-ax} \delta(\cos x)$$

(e)

$$\int_0^{\infty} dx \, \delta(e^{ax} \cos x).$$

(f)

$$\frac{d}{dx} e^{a\theta(x)}.$$

where a is a constant.