231 Tutorial Sheet 11: due Friday Febuary 3.[1,2]

27 January 2005

Useful facts:

- Parceval's formula:

$$\frac{1}{l}\int_{-l/2}^{l/2} dx|f(x)|^2 = \frac{1}{4}a_0^2+\frac{1}{2}\sum_{n=1}^{\infty}\left(a_n^2+b_n^2\right) = \sum_{n=-\infty}^{\infty}|c_n|^2$$

- The Fourier integral or Fourier transform:

$$f(x) = \int_{-\infty}^{\infty} dk\,\widetilde{f(k)}e^{ikx}$$
$$\widetilde{f(k)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx\,f(x)e^{-ikx}$$

- The Plancherel formula

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} dx\,|f(x)|^2 = \int_{-\infty}^{\infty} dk\,|\widetilde{f(k)}|^2$$

Questions

1. The Riemann zeta function is defined as follows

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s} \qquad (s>1).$$

(a) By applying Parseval's theorem for Fourier series to the sawtooth $f(x)=x$ for $-\pi<x<\pi$ compute

$$\zeta(2)=\sum_{n=1}^{\infty}\frac{1}{n^2}.$$

(b) Consider the Fourier expansion of $f(x)=x^2$, $\quad -\pi<x<\pi$, and use the result to show that

$$\zeta(4)=\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{\pi^4}{90}.$$

[1]Conor Houghton, `houghton@maths.tcd.ie`, see also `http://www.maths.tcd.ie/~houghton/231`

[2]Including material from Chris Ford, to whom many thanks.

Remark:

It is straightforward to compute $\zeta(n)$ if n is an even integer. On the other hand, no simple formulae are available for the odd integers, i.e. $\zeta(3)$, $\zeta(5)$, etc. Though, of course, they can be computed numerically.

The definition of $\zeta(s)$ straightforwardly extends to complex values of s. The sum defining $\zeta(s)$ clearly converges for any complex s where Re $s>1$. In fact, the function can be 'analytically continued' to *any* complex s (apart from $s=1$). Such a continuation boils down to finding an alternative expression for $\zeta(s)$ that agrees with the given definition when Re$s>1$ but makes sense for any $s\neq 1$. It is rather easy to continue to s values such that Re $s>0$. One recasts the zeta function as an alternating series

$$\zeta(s)=\frac{1}{1-2^{1-s}}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^s}.$$

A long standing problem in mathematics is to find the zeros of $\zeta(s)$. It is known that $\zeta(s)$ has zeros at the points $s=-2n$ where n is a natural number and that all other zeros lie in the strip $0\le$ Re $s\le 1$.

Riemann's hypothesis: All zeros of $\zeta(s)$ in the strip $0\le$ Re $s\le 1$ lie on the line Re $s=\frac{1}{2}$.

2. Compute the Fourier transform of $f(x)=e^{-a|x|}$ where a is a positive constant. Use the result to show that

$$\int_{-\infty}^{\infty} dp\,\frac{\cos p}{1+p^2}=\frac{\pi}{e}.$$

3. Determine the Fourier transform of the Gaussian function

$$f(x)=e^{-\alpha x^2},$$

where α is a positive constant.

Suggestions: Compute the integral

$$\int_{-\infty}^{\infty} dx\; e^{-\alpha x^2+\beta x},$$

where β is real. To do this, first consider the case $\beta=0$. Infer the result for $\beta\neq 0$ by a simple change of variable in the integral. Then formally allow β to be imaginary (you may wish to postpone the justification of this procedure to a later date!).