231 Tutorial Sheet 11.¹²

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Useful facts:

- A function f(x) has period l if f(x+l) = f(x), it is odd if f(-x) = -f(x) and even if f(-x) = f(x).
- A function with period l has the Fourier series expansion

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nx}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi nx}{l}\right).$$

where

$$a_0 = \frac{2}{l} \int_{-l/2}^{l/2} f(x) dx$$

$$a_n = \frac{2}{l} \int_{-l/2}^{l/2} f(x) \cos\left(\frac{2\pi nx}{l}\right) dx$$

$$b_n = \frac{2}{l} \int_{-l/2}^{l/2} f(x) \sin\left(\frac{2\pi nx}{l}\right) dx$$

• A function with period l has the Fourier series expansion

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \exp \frac{2\pi nx}{l}.$$

where

$$c_n = \frac{1}{l} \int_{-l/2}^{l/2} f(x) \exp\left(\frac{-2\pi nx}{l}\right) dx$$

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Questions

1. Express the following periodic functions $(l = 2\pi)$ as complex Fourier series

(a)

$$f(x) = \begin{cases} 0 & -\pi < x < -a \\ 1 & -a < x < a \\ 0 & a < x < \pi \end{cases}$$

where $a \in (0, \pi)$ is a constant.

(b)

$$f(x) = \frac{1}{2 - e^{ix}}$$

2. Show that the periodic function f defined by $f(x) = |x| - \frac{1}{2}\pi$ for $-\pi < x < \pi$ and $f(x+2\pi) = f(x)$ has the Fourier series expansion

$$f(x) = -\frac{4}{\pi} \sum_{n>0, \text{ odd}} \frac{\cos nx}{n^2}.$$

3. Use the Fourier series given in question 2 to compute the following sums

$$S_1 = 1 - \frac{1}{3^2} - \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} - \frac{1}{11^2} - \frac{1}{13^2} + \dots$$
$$S_2 = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \dots$$

Remark: With calculations of this kind it makes sense to try a quick numerical check of your answer.