231 Tutorial Sheet 10.[1][2]

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Useful facts:

- A function $f(x)$ has period l if $f(x+l)=f(x)$, it is odd if $f(-x)=-f(x)$ and even if $f(-x)=f(x)$.
- A function with period l has the Fourier series expansion

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos\left(\frac{2\pi nx}{l}\right)+\sum_{n=1}^{\infty} b_n\sin\left(\frac{2\pi nx}{l}\right).$$

where

$$\begin{aligned} a_0 &= \frac{2}{l}\int_{-l/2}^{l/2} f(x)dx \\ a_n &= \frac{2}{l}\int_{-l/2}^{l/2} f(x)\cos\left(\frac{2\pi nx}{l}\right)dx \\ b_n &= \frac{2}{l}\int_{-l/2}^{l/2} f(x)\sin\left(\frac{2\pi nx}{l}\right)dx \end{aligned}$$

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[2]Including material from Chris Ford, to whom many thanks.

Questions

1. Find the Fourier series representation of the sawtooth function f defined by $f(x) = x$ for $-\pi < x < \pi$ and $f(x + 2\pi) = f(x)$.

2. Establish that

$$\int_{-\pi}^{\pi} dx \ \sin mx \ \sin nx = \int_{-\pi}^{\pi} dx \ \cos mx \ \cos nx = 0,$$

   if $m \neq n$ (both m and n are integers).

3. The periodic function f is defined by

$$f(x) = \begin{cases} \sin x & 0 < x < \pi \\ 0 & -\pi < x < 0 \end{cases}$$

   and $f(x + 2\pi) = f(x)$.

   (a) Represent $f(x)$ as a Fourier series.

   Remarks: This function is neither odd nor even so both sets of Fourier coefficients are required. However it turns out that all the b_n are zero except for b_1. Can you see why this is the case without computing an integral?

   (b) Derive the remarkable formula

$$\frac{1}{2^2 - 1} + \frac{1}{4^2 - 1} + \frac{1}{6^2 - 1} + ... = \frac{1}{2}.$$