2007 Schol exam, three hours, do six. Outline solutions.

1. Find the volume of the region enclosed between the paraboloids

$$z = 5x^2 + 5y^2$$
$$z = 6 - 7x^2 - y^2$$

Solution: The volume is the integral of one; the awkward part is deciding what the limits of the integral are, for this, the z-part is easy, it is given, for the x and y limits we need the projection of the solid region down on to the xy-plane. By sketching the situation it is easy to see that the projection is given by the intersection, both parabolas are getting bigger as the spread out from the apices. Now the intersection is at

$$5x^2 + 5y^2 = 6 - 7x^2 - y^2 \tag{1}$$

which implies $2x^2 + y^2 = 1$, an ellipse, hence

$$\begin{aligned} V &= \int_{-1/\sqrt{2}}^{1/\sqrt{2}} dx \int_{-\sqrt{1-2x^2}}^{\sqrt{1-2x^2}} dy \int_{5x^2+5y^2}^{6-7x^2-y^2} dz \\ &= \int_{-1/\sqrt{2}}^{1/\sqrt{2}} dx \int_{-\sqrt{1-2x^2}}^{\sqrt{1-2x^2}} dy(6 - 12x^2 - 6y^2) \\ &= \int_{-1/\sqrt{2}}^{1/\sqrt{2}} dx(1-2x^2)^{3/2} \end{aligned} \tag{2}$$

Now we need to do a cosine substitution for x

$$V = \frac{8}{\sqrt{2}} \int_{-\pi/2}^{\pi/2} d\theta cos^4\theta = \frac{3\pi}{\sqrt{2}} \tag{3}$$

2. (a) What is the Jacobian? In two-dimensions calculate the Jacobian

$$dxdy = Jdvdu$$

where x and y are the usual Cartesian coördinates and $x = u+v/2$ and $y = v$.

(b) Calculate

$$\int_0^2 dy \int_{y/2}^{(y+4)/2} dxy^3(2x-y)e^{2x-y}$$

(c) Convert to spherical coördinates and evaluate

$$\int_0^1 dx \int_0^{\sqrt{1-x^2}} dy \int_0^{\sqrt{1-x^2-y^2}} dz \frac{1}{1+x^2+y^2+z^2}$$

Solution: Now, for a two-dimensional integral, consider the change of variables from (x, y) to (u, v) related by

$$\begin{aligned} x &= x(u,v) \\ y &= y(u,v) \end{aligned} \tag{4}$$

then

$$\int_D dxdy\phi(x,y) = \int_D dudv\phi(x(u,v), y(u,v))J \tag{5}$$

where the **Jacobian** J is the absolute value of the determinant

$$J = \left\| \begin{array}{cc} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{array} \right\| \tag{6}$$

For $x = u + v/2$ and $y = v$ the Jacobian is

$$J = \left\| \begin{array}{cc} 1 & 1/2 \\ 0 & 1 \end{array} \right\| = 1 \tag{7}$$

Now for the integral

$$\int_0^2 dy \int_{y/2}^{(y+4)/2} dxy^3(2x-y)e^{2x-y} = int_0^2 dv \int_0^2 duv^3ue^{2u} = 3e^4 + 1 \tag{8}$$

where integration by parts was used to do the u integral.

As for the second integral; as suggested, change to spherical polars

$$\begin{aligned} \int_0^{2\pi} d\phi \int_0^1 dr \int_0^{\pi/2} d\theta r^2 \sin\theta \frac{1}{1+r^2} &= 2\pi \int_0^1 dr \frac{r^2}{1+r^2} \\ &= 2\pi \left(1 - \frac{\pi}{2}\right) \end{aligned} \tag{9}$$

where the r integral can be done using a tan substitution.

3. State the Stokes theorem and give an outline proof which shows how the Stokes theorem can be reduced to Green's Theorem for vector fields of the form $\mathbf{F} = F_3(x, y, z)\mathbf{k}$ where S is of the form $z = h(x, y)$. Explain briefly how this special case is used to prove the Stokes theorem.

Solution: This is from the notes, see I.5.

4. For $\mathbf{r} = (x, y, z)$ the usual position vector show that

(a)
$$\operatorname{div}\operatorname{grad}\frac{1}{r} = 0$$

(b)
$$\operatorname{curl}\left[\mathbf{k}\times\operatorname{grad}\frac{1}{r}\right] + \operatorname{grad}\left[\mathbf{k}\cdot\operatorname{grad}\frac{1}{r}\right] = 0$$

(c) If $\mathbf{A}$ is a constant vector
$$\operatorname{grad}\left(\frac{\mathbf{A}\cdot\mathbf{r}}{r^3}\right) = \frac{\mathbf{A}}{r^3} - 3\frac{\mathbf{A}\cdot\mathbf{r}}{r^5}\mathbf{r}$$
where $r = |\mathbf{r}|$.

*Solution:*So, to begin
$$\begin{aligned}\operatorname{grad}\frac{1}{r} &= \frac{\partial}{\partial x}\frac{1}{r}\mathbf{i} + \frac{\partial}{\partial y}\frac{1}{r}\mathbf{j} + \frac{\partial}{\partial z}\frac{1}{r}\mathbf{k} \\ &= -\frac{x}{r^3}\mathbf{i} - \frac{y}{r^3}\mathbf{j} - \frac{z}{r^3}\mathbf{k}\end{aligned} \tag{10}$$
and hence
$$\operatorname{div}\operatorname{grad}\frac{1}{r} = -\frac{\partial}{\partial x}\frac{x}{r^3} - \frac{\partial}{\partial y}\frac{y}{r^3} - \frac{\partial}{\partial y}\frac{y}{r^3} = -\frac{3}{r^3} + 3\frac{x^2+y^2+z^2}{r^5} = 0 \tag{11}$$
as required. Next
$$\mathbf{k}\times\operatorname{grad}\frac{1}{r} = \mathbf{k}\times\left(-\frac{x}{r^3}\mathbf{i} - \frac{y}{r^3}\mathbf{j} - \frac{z}{r^3}\mathbf{k}\right) = \frac{y}{r^3}\mathbf{i} - \frac{x}{r^3}\mathbf{j} \tag{12}$$
so
$$\operatorname{curl}\left[\mathbf{k}\times\operatorname{grad}\frac{1}{r}\right] = \begin{pmatrix} -\frac{3xz}{r^5} \\ -\frac{3yz}{r^5} \\ \frac{1}{r^3} - \frac{3z^2}{r^5}\end{pmatrix} \tag{13}$$
and
$$\mathbf{k}\cdot\operatorname{grad}\frac{1}{r} = -\frac{z}{r^3} \tag{14}$$
so
$$\operatorname{grad}\left[\mathbf{k}\cdot\operatorname{grad}\frac{1}{r}\right] = \frac{3xz}{r^5}\mathbf{i} + \frac{3yz}{r^5}\mathbf{i} + \left(-\frac{1}{r^3} + \frac{3z^2}{r^5}\right)\mathbf{k} \tag{15}$$
as required.

5. Prove that for two periodic functions $f(x)$ and $g(x)$ with the same period l then
$$\frac{1}{l}\int_c^{c+l} dx f(x)g(x) = \int_{n=-\infty}^{\infty} c_n d_n^*$$
where c_n and d_n are the coefficients in the complex Fourier series for $f(x)$ and $g(x)$ respectively. Deduce Parseval's theorem from this.

*Solution:*Just substitute in
$$\frac{1}{l}\int_c^{c+l} dx f(x)g(x) = \frac{1}{l}\int_c^{c+l} dx \sum c_n e^{inxl/2\pi}\sum d_m e^{imxl/2\pi} \tag{16}$$
and then do a change of index to send m to $-m$ and use $d_{-m} = d_m^*$ to get
$$\frac{1}{l}\int_c^{c+l} dx f(x)g(x) = \sum_n\sum_m c_n d_m^* \frac{1}{l}\int_c^{c+l} dx e^{inxl/2\pi}e^{-imxl/2\pi} \tag{17}$$
and then the integral gives the required Kronecker δ-function. Setting $f = g$ gives Parseval's theorem.

6. Calculate the Fourier transform, $\tilde{f}(k)$, for the following pulses.

(a) The rectangular pulse
$$f(x) = \begin{cases} A & |x| \le L \\ 0 & |x| > L\end{cases}$$
where A is a constant.

(b) The two-sided exponential pulse
$$f(x) = \begin{cases} e^{ax} & |x| \le 0 \\ e^{-ax} & x > 0\end{cases}$$
where $a > 0$ is a constant.

*Solution:*The rectangular pulse is just a matter of integration
$$\tilde{f}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx\, f(x)e^{-ikx} = \frac{A}{2\pi}\int_{-L}^{L} dx e^{-ikx} = \frac{A\sin kL}{\pi} \tag{18}$$
The two-sided exponential is 07-08 PS11 q4.

7. The function $\phi(x)$ is monotone increasing in $[a, b]$ and has a zero at $x = c$ where $a < c < b$ and $\phi'(c) \neq 0$. Show that

$$\int_a^b f(x)\delta[\phi(x)]dx = \frac{f(c)}{\phi'(c)}$$

Show that the same formula applies if $\phi(x)$ is monotone decreasing and hence derive a formula for general $\phi(x)$ provided the zeros are simple. Deduce that

$$\delta(ax) = \frac{\delta(t)}{|a|}$$

for $a \neq 0$. Also establish that

$$\int_{-\infty}^{\infty} |x|\delta(x^2 - a^2) = 1$$

*Solution:*Most of this question is from the notes, it is II.3. The bit at the end is new; there are two roots of $x^2 - a^2$; $x = \pm a$ and at each the slope has absolute value $2a$ hence

$$\delta(x^2 - a^2) = \frac{1}{2a}[\delta(x - a) + \delta(x + a)] \tag{19}$$

and $\int_{-\infty}^{\infty} |x|\delta(x - a) = \int_{-\infty}^{\infty} |x|\delta(x + a) = a$ giving the result.

8. Find the general solution to

$$y''' - 3y'' + 3y' - y = e^x$$

*Solution:*So the auxillary equation is

$$\lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0 \tag{20}$$

giving $\lambda = 1$ three times. Hence, you might guess that the solution to the homogeneous equation is

$$y = C_1 e^x + C_2 x e^x + C_3 x^2 e^x \tag{21}$$

and this can be confirmed by just substituting in. For the inhomogenous equation try $y = Cx^3e^x$, substituting in gives

$$6C = 1 \tag{22}$$

so $C = 1/6$.