2008 Schol exam, three hours, do six.

1. Find the volume of the region enclosed between the paraboloids

$$\begin{aligned} z &= 5x^2 + 5y^2 \\ z &= 6 - 7x^2 - y^2 \end{aligned}$$

2. (a) What is the Jacobian? In two-dimensions calculate the Jacobian

$$dxdy = Jdvdu$$

where x and y are the usual Cartesian coördinates and $x = u + v/2$ and $y = v$.

(b) Calculate

$$\int_0^2 dy \int_{y/2}^{(y+4)/2} dx y^3 (2x - y) e^{2x-y}$$

(c) Convert to spherical coördinates and evaluate

$$\int_0^1 dx \int_0^{\sqrt{1-x^2}} dy \int_0^{\sqrt{1-x^2-y^2}} dz \frac{1}{1 + x^2 + y^2 + z^2}$$

3. State the Stokes theorem and give an outline proof which shows how the Stokes theorem can be reduced to Green's Theorem for vector fields of the form $\mathbf{F} = F_3(x, y, z)\mathbf{k}$ where S is of the form $z = h(x, y)$. Explain briefly how this special case is used to prove the Stokes theorem.

4. For $\mathbf{r} = (x, y, z)$ the usual position vector show that

(a)

$$\text{div}\,\text{grad}\,\frac{1}{r} = 0$$

(b)

$$\text{curl}\left[\mathbf{k} \times \text{grad}\,\frac{1}{r}\right] + \text{grad}\left[\mathbf{k} \cdot \text{grad}\,\frac{1}{r}\right] = 0$$

(c) If $\mathbf{A}$ is a constant vector

$$\text{grad}\left(\frac{\mathbf{A} \cdot \mathbf{r}}{r^3}\right) = \frac{\mathbf{A}}{r^3} - 3\frac{\mathbf{A} \cdot \mathbf{r}}{r^5}\mathbf{r}$$

where $r = |\mathbf{r}|$.

5. Prove that for two periodic functions $f(x)$ and $g(x)$ with the same period l then

$$\frac{1}{l}\int_{c}^{c+l} dx f(x)g(x) = \int_{n=-\infty}^{\infty} c_n d_n^*$$

where c_n and d_n are the coefficients in the complex Fourier series for $f(x)$ and $g(x)$ respectively. Deduce Parseval's theorem from this.

6. Calculate the Fourier transform, $\tilde{f}(k)$, for the following pulses.

   (a) The rectangular pulse

   $$f(x) = \begin{cases} A & |x| \leq L \\ 0 & |x| > L \end{cases}$$

   where A is a constant.

   (b) The two-sided exponenetial pulse

   $$f(x) = \begin{cases} e^{ax} & x \leq 0 \\ e^{-ax} & x > 0 \end{cases}$$

   where $a > 0$ is a constant.

7. The function $\phi(x)$ is monotone increasing in $[a, b]$ and has a zero at $x = c$ where $a < c < b$ and $\phi'(c) \neq 0$. Show that

$$\int_{a}^{b} f(x)\delta[\phi(x)]dx = \frac{f(c)}{\phi'(c)}$$

Show that the same formula applies if $\phi(x)$ is monotone decreasing and hence derive a formula for general $\phi(x)$ provided the zeros are simple. Deduce that

$$\delta(ax) = \frac{\delta(t)}{|a|}$$

for $a \neq 0$. Also establish that

$$\int_{-\infty}^{\infty} |x|\delta(x^2 - a^2) = 1$$

8. Find the general solution to

$$y''' - 3y'' + 3y' - y = e^x$$