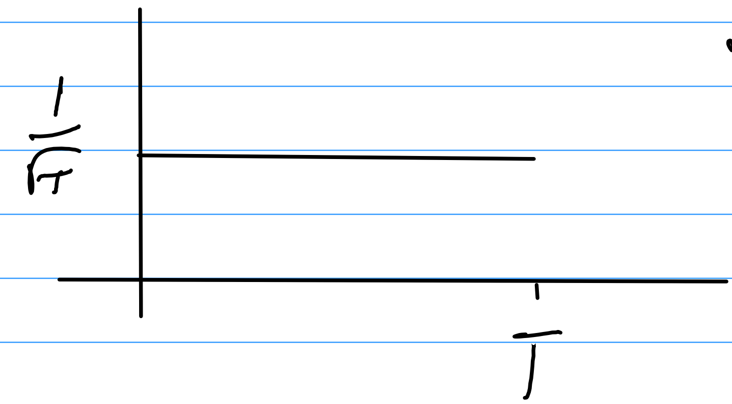
21 May 07

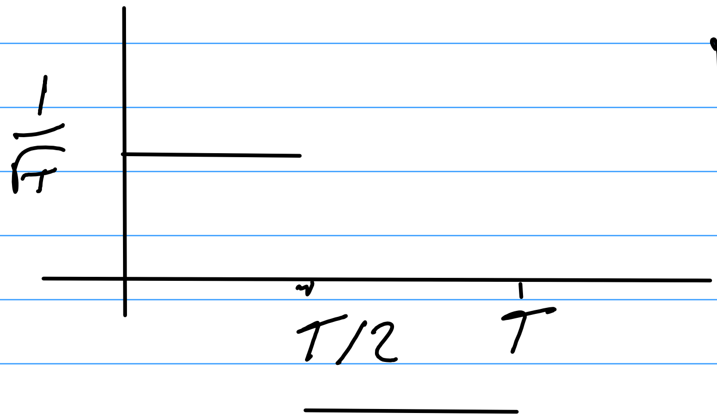
231 Schol 2007 <u>OUTLINE SOLNS</u>

Q5

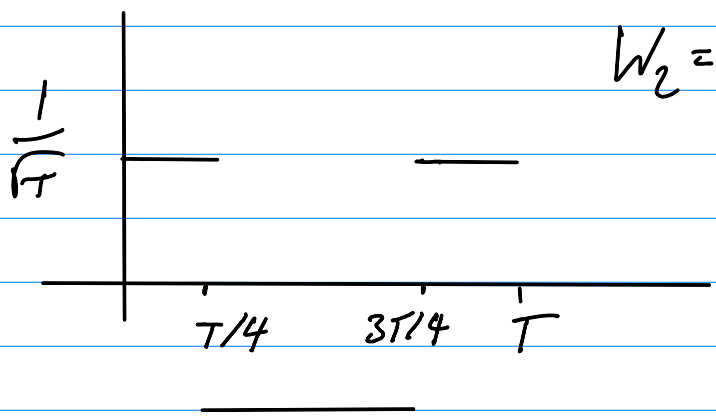
(a)



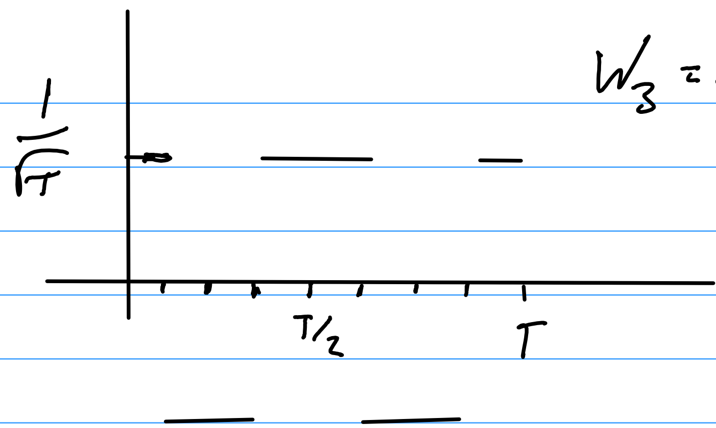
$$W_0 = \frac{1}{\sqrt{T}}$$



$$W_1 = \begin{cases} \frac{1}{\sqrt{T}} & t \in [0, \frac{T}{2}) \\ -\frac{1}{\sqrt{T}} & t \in (\frac{T}{2}, T] \end{cases}$$



$$W_2 = \begin{cases} \frac{1}{\sqrt{T}} & [0, \frac{T}{4}) \cup (\frac{3T}{4}, T] \\ -\frac{1}{\sqrt{T}} & (\frac{T}{4}, \frac{3T}{4}) \end{cases}$$



$$W_3 = \begin{cases} \frac{1}{\sqrt{T}} & [0, \frac{T}{8}) \cup (\frac{3T}{8}, \frac{5T}{8}) \cup (\frac{7T}{8}, T] \\ -\frac{1}{T} & (\frac{T}{8}, \frac{3T}{8}) \cup (\frac{5T}{8}, \frac{7T}{8}) \end{cases}$$

rather than do the individual orthogalizations by hand lets go straight to (b) & do them all in one go!

(b)

$W_1(t)$

$$w_2(t) = \begin{cases} W_1(2t) & t < \frac{T}{2} \\ -W_1(2t) & t > \frac{T}{2} \end{cases}$$

$$W_3(t) = \left\{ (-1)^{\alpha(n)} W(4t) \quad \left(\frac{T}{4}\right)^n < t < \left(\frac{T}{4}\right)^{n+1} \right.$$

so

$$W_i = \left\{ (-1)^{\alpha(n)} W((i+1)t) \quad \left(\frac{T}{i+1}\right)^n < t < \left(\frac{T}{i+1}\right)^{n+1} \right.$$

where $\alpha(n)$ is one or zero depending on n.

now $\int_0^T W_i \, dt = 0$ ($i>0$) since $\int_0^T W_1(t) \, dt = 0$

& $\int_0^T W_0 W_i \, dt = \frac{1}{\sqrt{T}} \int_0^T W_i \, dt = 0$

$$W_i W_j = \left\{ (-1)^{\beta(n)} W((j+1)t) \quad \left(\frac{T}{j+1}\right)^n < t < \left(\frac{T}{j+1}\right)^{n+1} \right.$$

where wlog $j>i$ & $\beta(n)$ is zero or one. This is because small subinterval fits into at most half of the larger subinterval

$\Rightarrow \int_0^T W_i W_j \, dt = 0 \qquad \int W_i^2 dt = \frac{1}{\sqrt{T}} \int_0^T W_0 \, dt = 1$

finally

$$f(t) = \sum c_a W_a(t)$$

$$\int_0^T W_b f(t) \, dt = \sum c_a \int_0^T W_a(t) W_b(t) \, dt$$

$$= c_b$$

Q7 $$f * g(x) = \int_{-\infty}^{\infty} d\sigma\, f(\sigma)\, g(x-\sigma)$$

$$\tilde{h}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx\, h(x)\, e^{-ikx}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} d\sigma\, f(\sigma)\, g(x-\sigma)\, e^{-ikx}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} dx\, d\sigma\, f(\sigma) g(x-\sigma)\, e^{-ik\sigma} e^{-ik(x-\sigma)}$$

change variable to $\sigma' = x - \sigma$

$$= 2\pi \frac{1}{2\pi} \int_{-\infty}^{\infty} d\sigma'\, g(\sigma')\, e^{-ik\sigma'} \frac{1}{2\pi} \int d\sigma\, f(\sigma)\, e^{-ik\sigma}$$

$$= 2\pi\, \tilde{g}(k)\, \tilde{f}(k)$$ as required

Q8

$$x^2y'' + 3xy' + y = 0$$

$x = e^u \quad u = \log x$

$$y' = \frac{dy}{dx} = \frac{du}{dx}\frac{dy}{du}$$
$$= \frac{1}{x}\frac{dy}{du}$$

$$y'' = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{du}\right)$$
$$= -\frac{1}{x^2}\frac{dy}{du} + \frac{1}{x^2}\frac{d^2y}{du^2}$$

$$\frac{d^2y}{du^2} + 2\frac{dy}{du} + y = 0$$

$(\lambda^2 + 2\lambda + 1) = 0 \qquad y = e^{\lambda u}$

$\lambda = -1$ twice

$$\Rightarrow \quad y = C_1 e^{-u} + C_2 u e^{-u}$$
$$= \frac{C_1}{x} + \frac{C_2 \log x}{x}$$

$$x^2y'' + 3xy' + y = 0$$

check since it looks improbable

$$y = \frac{\log x}{x}$$

$$y' = \frac{1 - \log x}{x^2}$$

$$y'' = \frac{-2}{x^3} - \frac{x - 2x\log x}{x^4}$$

$$\Rightarrow$$

$$\frac{-3}{x} + \frac{2\log x}{x^2} + \frac{3}{x} - \frac{3\log x}{x^2} + \frac{\log x}{x} = 0$$