2007 School <u>OUTLINE SOLN</u>

17 May 2007

Q3

let D be a smooth region in $\mathbb{R}^3$, with boundary δD, let $\underline{F}$ be a smooth vector field defined in a nbd of D, then

$$\int_D dV \, \mathrm{div}\, \underline{F} = \int_{\delta} \underline{F} \cdot \underline{dS}$$

let $\underline{F} = \phi \underline{\nabla} \psi - \psi \underline{\nabla} \phi$

$$\begin{aligned} \mathrm{div}\, \underline{F} &= \underline{\nabla} (\phi \underline{\nabla} \psi - \psi \underline{\nabla} \phi) \\ &= \cancel{\underline{\nabla} \phi \cdot \underline{\nabla} \psi} + \phi \Delta \psi - \cancel{\underline{\nabla} \psi \cdot \underline{\nabla} \phi} \\ &\quad - \psi \Delta \phi = \phi \Delta \psi - \psi \Delta \phi \end{aligned}$$

hence $\displaystyle\int_V \mathrm{div}\, F \, dV = \int_{\delta V} \underline{F} \cdot \underline{dS}$

$$\Rightarrow \iiint_V dV \, (\phi \Delta \psi - \psi \Delta \phi) = \iint_{\delta V} [\phi \underline{\nabla} \psi - \psi \underline{\nabla} \phi] \cdot \underline{dS}$$

$\underset{\sim}{A} = -2x^2\underline{i} + 4y\underline{j} + z^2\underline{k}$

by Gauß

$$\operatorname{div}\underline{A} = -4x + 4 + 2z$$

$$\int_{\delta V} \underline{A}\cdot d\underline{S} = \int_V \operatorname{div}\underline{A}\, dV$$

$$= \int_V (-4x + 4 + 2z)\, dV$$

use cylindrical polars

$$= \int_0^1 d\rho \int_0^1 dz \int_0^{2\pi} d\phi\, (-4\rho\cos\phi + 4 + 2z)$$

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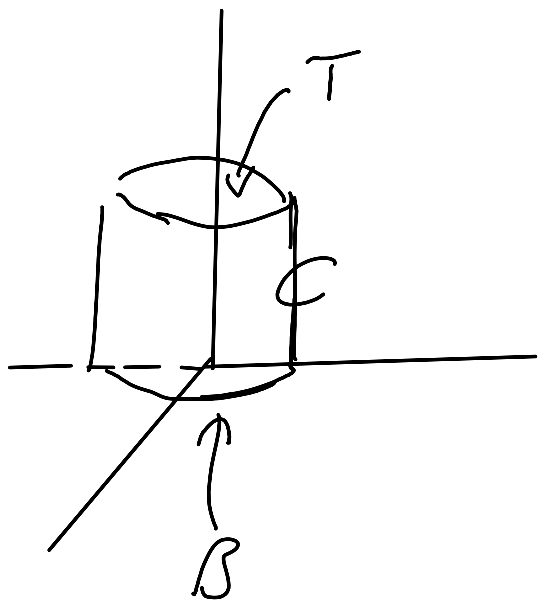
$$= 5\pi$$



directly

$$S = C + T + B$$

$\underline{B}$ $\quad \hat{\underline{n}} = -\hat{k}$

$\underline{A} \cdot \underline{\hat{n}} = -z^2$ which is zero on B

$\underline{T}$ $\underline{\hat{n}} = \underline{\hat{k}}$ & $\underline{A} \cdot \underline{\hat{n}}\Big|_T = z^2 = 1$

hence $\int_T \underline{A} \cdot d\underline{S} = \int_T dA = \pi$

$\underline{C}$ $\underline{n} = \cos\phi \, \underline{i} + \sin\phi \, \underline{j}$

$$\underline{A} \cdot \underline{n}\Big|_C = -2x^2 \cos\phi + 4y \sin\phi$$

$$= -2\cos^3\phi + 4\sin^2\phi$$

& $d\underline{S} = \underline{n} \, dz \, d\phi$

derive from

$$d\underline{S} = \frac{d\underline{r}}{dz} \times \frac{d\underline{r}}{d\phi}$$

hence

$$\int_C \underline{A} \cdot d\underline{S} = \int_0^{2\pi} d\phi \int_0^1 dz \left(4\sin^2\phi \underbrace{- 2\cos^3\phi}\right)$$

integrates to zero since odd.

$$\ldots = 4\pi$$

hence

$$\int_S \underline{A} \cdot d\underline{S} = \pi + 4\pi = 5\pi$$