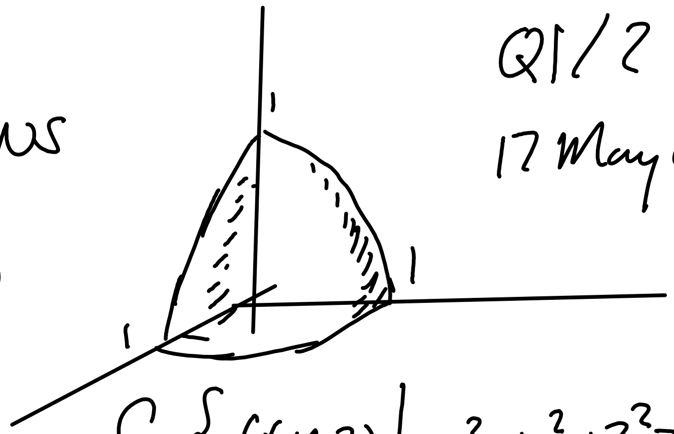
Schol 2007 <u>OUTLINE</u> SOLNS Q1/2 17 May 07

Q1

$$\int_S dS\,(x+y+z)$$



$$S = \{ (x,y,z) \mid x^2+y^2+z^2 = 1 \;\&\; x,y,z \text{ all } +ve \}$$

from geometry of S should change to spherical polars

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

$r = 1$ on S

$$dS = \left| \frac{d\underline{r}}{d\theta} \times \frac{d\underline{r}}{d\phi} \right|$$

where $\underline{r} = \sin\theta\cos\phi\,\underline{i} + \sin\theta\sin\phi\,\underline{j} + \cos\theta\,\underline{k}$

& the usual calculation gives

$$dS = \sin\theta\, d\theta\, d\phi$$

hence

$$\int_S dS\,(x+y+z) = \int_S dS\,(\sin\theta\cos\phi + \sin\theta\sin\phi + \cos\theta)$$

$$= \int_0^{\pi/2} d\theta \int_0^{\pi/2} d\phi\,\left(\sin^2\theta\cos\phi + \sin^2\theta\sin\phi + \sin\theta\cos\theta\right)$$

$$= \int_0^{\pi/2} d\theta\,\left(2\sin^2\theta + \frac{\pi}{2}\sin\theta\cos\theta\right)$$

$$= \frac{3\pi}{4}$$

Q2

a) In a change of coördinates the <u>Jacobian</u> relates the area elements

$$dx_1\,dx_2\,...\,dx_n = J\,du_1\,du_2\,...\,du_n$$

$$J = \frac{\partial(x_1, ..., x_n)}{\partial(u_1, ..., u_n)}$$

$$= \det\begin{pmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_2}{\partial u_1} & \cdots & \frac{\partial x_n}{\partial u_1} \\ \frac{\partial x_1}{\partial u_2} & \cdots & & \frac{\partial x_n}{\partial u_2} \\ \vdots & & \ddots & \vdots \\ \frac{\partial x_1}{\partial u_n} & \cdots & & \frac{\partial x_n}{\partial u_n} \end{pmatrix}$$

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} = |r| = r$$

$$dx\,dy = J\,d\theta\,dr$$

b) $$\int_{-\infty}^{\infty} dx\, e^{-ax^2 + bx - c}$$

$$= \int_{-\infty}^{\infty} dx\; e^{-(\sqrt{a}x - \frac{b}{2\sqrt{a}})^2 + \frac{b^2}{4a} - c}$$

$$= e^{\frac{b^2}{4a} - c} \int_{-\infty}^{\infty} dx\; e^{-(\sqrt{a}x - \frac{b}{2\sqrt{a}})^2}$$

$$u = \sqrt{a}x - \frac{b}{2\sqrt{a}}$$

$$du = \sqrt{a}\,dx.$$

$$= \frac{1}{\sqrt{a}} e^{\frac{b^2}{4a} - c} \underbrace{\int_{-\infty}^{\infty} du\, e^{-u^2}}$$

do the usual argument here

$$= \sqrt{\pi/a}\; e^{b^2/4a - c} \int_{-\infty}^{\infty} du\, e^{-u^2}$$