2007 Schol exam, three hours, do six.

1. Calculate

$$\int_{S} dS(x+y+z)$$

for S the portion of the sphere $x^2 + y^2 + z^2 = 1$ where x, y and z are all positive.

2. (a) What is the Jacobian? In two-dimensions calculate the Jacobian

$$dxdy = Jd\theta dr$$

where x and y are the usual Cartesian coördinates and r and θ the usual polar coördinates.

(b) Calculate

$$\int_{-\infty}^{\infty} dx \ e^{-ax^2 + bx - c} = \sqrt{\pi}$$

where a, b and c are real constants and a is positive.

3. State carefully the Gauß divergence theorem. Use it to show

$$\int \int \int_{V} dV \left[\phi \triangle \psi - \psi \triangle \phi \right] = \int \int_{\delta V} \left[\phi \nabla \psi - \psi \nabla \phi \right] \cdot bf dS$$

For $\mathbf{A} = -2x^2\mathbf{i} + 4y\mathbf{j} + z^2\mathbf{k}$ evaluate $\int_S \mathbf{A} \cdot \mathbf{dS}$ where S is the cylinder $x^2 + y^2 = 1$ with $0 \le z \le 1$ and $x^2 + y^2 \le 1$ with z = 0 or z = 1, that is, the cylinder with both ends included.

4. Show that for a vector field \mathbf{F}

$$\operatorname{curl}\operatorname{curl}\mathbf{F} = \operatorname{grad}\operatorname{div}\mathbf{F} - \bigtriangleup\mathbf{F}$$

Show that the equations

$$div \mathbf{H} = 0$$

$$div \mathbf{E} = 0$$

$$curl \mathbf{H} = \frac{\partial \mathbf{E}}{\partial t}$$

$$curl \mathbf{E} = -\frac{\partial \mathbf{H}}{\partial t}$$

for the time dependent vector fields $\mathbf{H}(\mathbf{x}, t)$ and $\mathbf{E}(\mathbf{x}, t)$ are satisfied by

where $\mathbf{Z}(\mathbf{x}, t)$ satisfies

$$\triangle \mathbf{Z} = \frac{\partial^2 \mathbf{Z}}{\partial t^2}.$$

You may quote the formulas $\operatorname{curl}\operatorname{grad}\phi = 0$ for scalar field ϕ and div curl, $\mathbf{F} = 0$ for vector field \mathbf{F} .

5. The Fourier series relies on the orthogonality of the sine and cosine functions; other, similar expansions can be defined using other sets of orthonormal functions. In digital signal processing the Walsh functions $W_n(t)$ are commonly used. On the interval [0, T] the first four Walsh functions are

$$W_{0}(t) = \frac{1}{\sqrt{T}}$$

$$W_{1}(t) = \begin{cases} 1/\sqrt{T} & t \in [0, T/2) \\ -1/\sqrt{T} & t \in (T/2, T] \end{cases}$$

$$W_{2}(t) = \begin{cases} 1/\sqrt{T} & t \in [0, T/4) \cup (3T/4, T] \\ -1/\sqrt{T} & t \in (T/4, 3T/4) \end{cases}$$

$$W_{3}(t) = \begin{cases} 1/\sqrt{T} & t \in [0, T/8) \cup (3T/8, 5T/8) \cup (7T/8, T] \\ -1/\sqrt{T} & t \in (T/8, 3T/8) \cup (5T/8, 7T/8) \end{cases} (1)$$

(a) Plot graphs of the functions W_0 , W_1 , W_2 and W_3 and demonstrate the orthonormality relation

$$\int_0^T dt W_a(t) W_b(t) = \delta_{ab}$$

for a and b less than four.

(b) Define $W_a(t)$ so that they satisfy orthonomality relation for all a. Find c_a for the Fourier-Walsh expansion of a function f(t):

$$f(t) = \sum_{a=0}^{\infty} c_a W_a(t)$$

6. Write the triangular pulse

$$f(t) = \begin{cases} At/T + A & -T < t < 0\\ -At/T + A & 0 < t < T\\ 0 & |t| > T \end{cases}$$

as a Fourier integral.

7. The convolution of two functions f(x) and g(x) is defined as

$$f \star g(x) = \int_{-\infty}^{\infty} d\sigma f(\sigma) g(x - \sigma)$$
(2)

If $h(x) = f \star g(x)$ show that the Fourier coefficient

$$\widetilde{h(k)} = 2\pi \widetilde{f(k)} \widetilde{g(k)}$$
(3)

where $\widetilde{f(k)}$ and $\widetilde{g(k)}$ are the Fourier coefficients of f(x) and g(x).

8. Solve the Euler-Cauchy equation

$$x^2y'' + 3xy' + y = 0.$$