2007 Schol exam, three hours, do six.

1. Calculate
$$\int_S dS(x+y+z)$$
for S the portion of the sphere $x^2+y^2+z^2=1$ where x, y and z are all positive.

2. (a) What is the Jacobian? In two-dimensions calculate the Jacobian
$$dxdy = Jd\theta dr$$
where x and y are the usual Cartesian coördinates and r and θ the usual polar coördinates.

   (b) Calculate
$$\int_{-\infty}^{\infty} dx\; e^{-ax^2+bx-c} = \sqrt{\pi}$$
where a, b and c are real constants and a is positive.

3. State carefully the Gauß divergence theorem. Use it to show
$$\int\int\int_V dV\,[\phi\triangle\psi - \psi\triangle\phi] = \int\int_{\delta V} [\phi\nabla\psi - \psi\nabla\phi]\cdot bf dS$$
For $\mathbf{A} = -2x^2\mathbf{i} + 4y\mathbf{j} + z^2\mathbf{k}$ evaluate $\int_S \mathbf{A}\cdot\mathbf{dS}$ where S is the cylinder $x^2+y^2=1$ with $0\le z\le 1$ and $x^2+y^2\le 1$ with $z=0$ or $z=1$, that is, the cylinder with both ends included.

4. Show that for a vector field $\mathbf{F}$
$$\mathrm{curl}\,\mathrm{curl}\,\mathbf{F} = \mathrm{grad}\,\mathrm{div}\,\mathbf{F} - \triangle\mathbf{F}$$
Show that the equations
$$\begin{aligned}\mathrm{div}\,\mathbf{H} &= 0\\ \mathrm{div}\,\mathbf{E} &= 0\\ \mathrm{curl}\,\mathbf{H} &= \frac{\partial\mathbf{E}}{\partial t}\\ \mathrm{curl}\,\mathbf{E} &= -\frac{\partial\mathbf{H}}{\partial t}\end{aligned}$$

for the time dependent vector fields $\mathbf{H}(\mathbf{x},t)$ and $\mathbf{E}(\mathbf{x},t)$ are satisfied by
$$\begin{aligned}\mathbf{H} &= \mathrm{curl}\,\frac{\partial\mathbf{Z}}{\partial t}\\ \mathbf{E} &= \mathrm{curl}\;\mathrm{curl}\mathbf{Z}\end{aligned}$$
where $\mathbf{Z}(\mathbf{x},t)$ satisfies
$$\triangle\mathbf{Z} = \frac{\partial^2\mathbf{Z}}{\partial t^2}.$$
You may quote the formulas $\mathrm{curl}\,\mathrm{grad}\,\phi = 0$ for scalar field ϕ and $\mathrm{div}\,\mathrm{curl},\mathbf{F} = 0$ for vector field $\mathbf{F}$.

5. The Fourier series relies on the orthogonality of the sine and cosine functions; other, similar expansions can be defined using other sets of orthonormal functions. In digital signal processing the Walsh functions $W_n(t)$ are commonly used. On the interval $[0,T]$ the first four Walsh functions are
$$\begin{aligned}W_0(t) &= \frac{1}{\sqrt{T}}\\ W_1(t) &= \begin{cases}1/\sqrt{T} & t\in[0,T/2)\\ -1/sqrtT & t\in(T/2,T]\end{cases}\\ W_2(t) &= \begin{cases}1/\sqrt{T} & t\in[0,T/4)\cup(3T/4,T]\\ -1/sqrtT & t\in(T/4,3T/4)\end{cases}\\ W_3(t) &= \begin{cases}1/\sqrt{T} & t\in[0,T/8)\cup(3T/8,5T/8)\cup(7T/8,T]\\ -1/sqrtT & t\in(T/8,3T/8)\cup(5T/8,7T/8)\end{cases}\end{aligned}\qquad(1)$$

   (a) Plot graphs of the functions W_0, W_1, W_2 and W_3 and demonstrate the orthonormality relation
$$\int_0^T dtW_a(t)W_b(t) = \delta_{ab}$$
for a and b less than four.

   (b) Define $W_a(t)$ so that they satisfy orthonomality relation for all a. Find c_a for the Fourier-Walsh expansion of a function $f(t)$:
$$f(t) = \sum_{a=0}^{\infty} c_a W_a(t)$$

6. Write the triangular pulse

$$f(x) = \begin{cases} At/T + A & -T < t < 0 \\ -At/T + A & 0 < t < T \\ 0 & |t| > T \end{cases}$$

as a Fourier integral.

7. The convolution of two functions $f(x)$ and $g(x)$ is defined as

$$f \star g(x) = \int_{-\infty}^{\infty} d\sigma f(\sigma) g(x - \sigma) \tag{2}$$

If $h(x) = f \star g(x)$ show that the Fourier coefficient

$$\widetilde{h(k)} = 2\pi \widetilde{f(k)} \widetilde{g(k)} \tag{3}$$

where $\widetilde{f(k)}$ and $\widetilde{g(k)}$ are the Fourier coefficients of $f(x)$ and $g(x)$.

8. Solve the Euler-Cauchy equation

$$x^2 y'' + 3xy' + y = 0.$$