231 FS 2006 q5/6 Outline Solution.[1]

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5. Ok so, it didn't say whether to take the real of the complex Fourier series: lets take the real. $|\cos x|$ is periodic with period π so

$$a_n = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} dx|\cos x|\cos nx = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} dx\cos x\cos nx \quad (1)$$

and,

$$\begin{aligned}\frac{2}{\pi}\int_{-\pi/2}^{\pi/2} dx\cos x\cos nx &= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} dx\left(e^{(n+1)ix} + e^{(-n+1)ix}\right)\\ &= \frac{1}{i\pi}\left[\frac{1}{n+1}e^{(n+1)ix} + \frac{1}{-n+1}e^{(-n+1)ix}\right]_{-\pi/2}^{\pi/2}\\ &= \frac{2}{\pi}\left(\frac{1}{n+1}\sin(n+1)\pi/2 + \frac{1}{-n+1}\sin(1-n)\pi/2\right) \quad (2)\end{aligned}$$

Now, for n odd $n\pm 1$ is even and so $\sin(n\pm 1)\pi/2$ is zero, for n even

$$\begin{aligned}\sin(n+1)\pi/2 &= \cos n\pi/2 = (-1)^{n/2}\\ \sin(-n+1)\pi/2 &= \cos n\pi/2 = (-1)^{n/2} \quad (3)\end{aligned}$$

since $cosm\pi = (-1)^m$ and n is of the form $2m$ if it is even. Thus, for n even

$$a_n = \frac{2}{\pi}\left(\frac{1}{n+1} + \frac{1}{-n+1}\right)(-1)^{n/2} = \frac{4}{1-n^2} \quad (4)$$

As for a_0

$$a_0 = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} dx|\cos x| = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} dx\cos x = \frac{2}{\pi}\sin x]_{-\pi/2}^{\pi/2} = \frac{4}{\pi} \quad (5)$$

Thus, changing n to $2n$,

$$|\cos x| = \frac{2}{\pi} + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{1-4n^2}\cos 2nx \quad (6)$$

6. Ok, so follow the definition

$$\mathcal{L}\left[e^{at}\right] = \int_0^{\infty} e^{at}e^{-st}dt = \int_0^{\infty} e^{-(s-a)t}dt = \frac{1}{s-a} \quad (7)$$

[1] Conor Houghton, houghton@maths.tcd.ie, see also http://www.maths.tcd.ie/~houghton/231

and

$$\mathcal{L}\left[f'(t)\right] = \int_0^\infty f'(t)e^{-st}dt = f(t)e^{-st}\Big]_{t=0}^{t=\infty} + s\int_0^\infty f(t)e^{-st}dt = -f(0) + s\mathcal{L}[f(t)] \tag{8}$$

and finally, taking the Laplace transform of the equation

$$sF - 1 = F \tag{9}$$

so $F = 1/(s-1)$ and hence $f(t) = \exp{(-t)}$.