231 FS 2007 q2/3 Outline Solution.[1]

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2. So the first part of this question is from the notes; as for the other part, we have

$$\mathbf{F} = 3x^2\mathbf{i} + 2yz\mathbf{j} + y^2\mathbf{k} \tag{1}$$

Actually, it is easy to spot this is grad of $\phi = x^3 + zy^2$, to actually calculate it if you don't spot it, we have

$$\frac{\partial}{\partial x}\phi = 3x^2 \tag{2}$$

so, integrting, $\phi = x^3 + A(y,z)$, now, from that

$$\frac{\partial}{\partial y}A(y,z) = 2yz \tag{3}$$

and hence $A(y,z) = y^2 z + B(z)$ and finally, substituting again

$$\frac{\partial}{\partial z}B(z) = 0 \tag{4}$$

so $B(z)$ is a constant, which, wlog, we choose as zero. Now $\phi(1,-1,7) - \phi(0,1,2) = 6$.

3. The cool part of this equation is the formula itself, where you can see the two different circles inside the torus in terms of the two angles u and v. The surface integral is

$$A = \int_S dS = \int_0^{2\pi} du \int_0^{2\pi} dv \left|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right| \tag{5}$$

From the question

$$\mathbf{r} = (a + b\cos v)\cos u\mathbf{i} + (a + b\cos v)\sin u\mathbf{j} + b\sin v\mathbf{k} \tag{6}$$

hence

$$\begin{aligned} \frac{\partial \mathbf{r}}{\partial u} &= -(a + b\cos v)\sin u\mathbf{i} + (a + b\cos v)\cos u\mathbf{j} \\ \frac{\partial \mathbf{r}}{\partial v} &= -b\sin v\cos u\mathbf{i} - b\sin v\sin u\mathbf{j} + b\cos v\mathbf{k} \end{aligned} \tag{7}$$

and hence, with shorthand for the cosines and sines

$$\begin{aligned} \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -(a+bc_2)s_1 & (a+bc_2)c_1 & 0 \\ -bs_2c_1 & -bs_1s_2 & bc_2 \end{vmatrix} \\ &= \begin{pmatrix} b(a+bc_2)c_1c_2 \\ -b(a+bc_2)s_1c_2 \\ b(a+bc_2)s_2 \end{pmatrix} \end{aligned} \tag{8}$$

and so taking the length

$$\left|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right|^2 = b^2(a+bc_2)^2 \tag{9}$$

where lots of simplifications have come from the Pythagorous theorem. Now the integral gives

$$A = 4\pi^2 ab \tag{10}$$

which is cool, it is the same as the surface area of a cylinder of radius a and length $2\pi b$!

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