231 FS 2007 q1 Outline Solution.[1]

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1(a) Ok, so it ins't so clear why its a good thing to do, but follow the instructions and do the change of variable: $x^2 + y^2 = (u^2 + v^2)/2$ either by solving for x and y or just by staring at it. Now, the Jacobian is

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{4}\left\|\begin{array}{cc} 1 & 1 \\ 1 & -1 \end{array}\right\| = \frac{1}{2} \tag{1}$$

and under the change of coördinates

$$\begin{array}{rcl} (0,0) & \rightarrow & (0,0) \\ (1,1) & \rightarrow & (2,0) \\ (2,0) & \rightarrow & (2,2) \\ (1,-1) & \rightarrow & (0,2) \end{array} \tag{2}$$

So, now I get it, the coördinate change makes the boundaries easier

$$\int\int_R (x^2+y^2)dxdy = \frac{1}{2}\int_0^2 du \int_0^2 dv(u^2+v^2) = \frac{1}{2}\int_0^2 du\left(2u^2+\frac{8}{3}\right) = \frac{8}{3} \tag{3}$$

1(b) So lots of repitition here, that was mostly a test of nerve, yes the x and y integrals are the same, being asked both makes you think they should be different, but they aren't. The area first, convert to polars, remembering the r from the Jacobian.

$$A = \int_0^{\pi/2} d\theta \int_0^1 drr = \frac{\pi}{4} \tag{4}$$

The centroids require we work out the first moments, in some made up notation:

$$< x >= \int_0^{\pi/2} d\theta \int_0^1 drxr = \int_0^{\pi/2} d\theta \int_0^1 drr^2\cos\theta = \frac{1}{3} \tag{5}$$

where the $1/3$ comes from the r integral and the one from the θ, and

$$< y >= \int_0^{\pi/2} d\theta \int_0^1 dryr = \int_0^{\pi/2} d\theta \int_0^1 drr^2\sin\theta = \frac{1}{3} \tag{6}$$

Hence $\bar{x} = \bar{y} = 4/3\pi$. Finally, the moments of inertia, I know the symbols I_{xx} and I_{yy} are completely misused here, that was corrected on the day, let me call them $< xx >$ and $< yy >$ instead.

$$< xx >= \int_0^{\pi/2} d\theta \int_0^1 drx^2r = \int_0^{\pi/2} d\theta \int_0^1 drr^3\cos^2\theta = \frac{1}{4}\int_0^{\pi/2}\cos^2\theta = \frac{\pi}{16} \tag{7}$$

using the usual $\cos^2\theta = (1+\cos 2\theta)/2$. $< yy >$ is the same.

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