UNIVERSITY OF DUBLIN

2130

TRINITY COLLEGE

FACULTY OF SCIENCE

SCHOOL OF MATHEMATICS

SF Mathematics
SF Theoretical Physics
Foundation Scholarship

Hilary Term 2006

COURSE 231

Friday, March 31 Exam Hall 09.30 — 12.30

Dr. Conor Houghton

Attempt SIX questions

Log tables are available from the invigilators, if required.

Non-programmable calculators are permitted for this examination,—please indicate the make and model of your calculator on each answer book used. There is a table of Fourier series and Fourier tranform formula after the questions.

1. (a) Use the change of variables $x + y = u$ and $x - y = v$ to evaluate

$$\int\int_R (x^2 + y^2) dx dy$$

where R is the square with corners $(0,0)$, $(1,1)$, $(2,0)$ and $(1,-1)$.

(b) Find the area and centroid for the quarter disk

$$R = \{(x,y) | x^2 + y^2 \leq 1, x \geq 0, y \geq 0\}$$

Find the moments of intertia:

$$\begin{aligned} I_{xx} &= \int\int_R y^2 dx dy \\ I_{yy} &= \int\int_R x^2 dx dy \end{aligned}$$

2. Define a conservative vector field and show a vector field is conservative if and only if its line integral is path independent. Evaluate the integral

$$I = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (3x^2 dx + 2yz dy + y^2 dz)$$

from $(0,1,2)$ to $(1,-1,7)$ by showing that $\mathbf{F}$ has a potential.

3. A torus can be obtained by rotating a circle of radius b around the z-axis, this gives a parameteric representation for the torus:

$$\mathbf{r} = (a + b\cos v)\cos u\mathbf{i} + (a + b\cos v)\sin u\mathbf{j} + b\sin v\mathbf{k}$$

where $a > b$ is the distance of the original circle from the z-axis and both u and v are angles in the range $[0, 2\pi)$. Find the surface area of a torus.

4. Using Gauss' theorem or otherwise compute the flux of the vector field $\mathbf{F} = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$ through the hemisphere $x^2 + y^2 + z^2 = 1$, $z \geq 0$ with the orientation taken upwards.

5. Express $f(x) = |\cos x|$ as a Fourier series.

6. The Laplace transform of a function $f(t)$, $t \geq 0$ is a function of s defined as

$$\mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-st}dt$$

Show

$$\mathcal{L}\left[e^{at}\right] = \frac{1}{s-a}$$

where a is a constant and you may assume $s > a$. Show that, if we define $F(s)$ as

$$\mathcal{L}[f(t)] = F(s)$$

then, by integrating by parts

$$\mathcal{L}\left[\frac{df}{dt}\right] = sF(s) - f(0)$$

where you can assume

$$\lim_{t\to\infty} f(t)e^{-st} = 0$$

By taking the Laplace transform of both sides of the equation solve

$$\frac{df}{dt} = f$$

with $f(0) = 1$.

7. Determine the Fourier integral of the Gaussian function

$$f = e^{-ax^2}$$

8. Assuming there is a series solution

$$y = \sum_{n=0}^{\infty} a_n x^n$$

to

$$y'' - 3x^2 y = 0$$

with $y(0) = 1$ and $y'(0) = -1$ write down the first four non-zero terms of the series.

Some useful formula

- A function with period l has the Fourier series expansion

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n x}{l}\right).$$

where

$$\begin{aligned} a_0 &= \frac{2}{l}\int_{-l/2}^{l/2} f(x)dx \\ a_n &= \frac{2}{l}\int_{-l/2}^{l/2} f(x)\cos\left(\frac{2\pi n x}{l}\right)dx \\ b_n &= \frac{2}{l}\int_{-l/2}^{l/2} f(x)\sin\left(\frac{2\pi n x}{l}\right)dx \end{aligned}$$

- A function with period l has the Fourier series expansion

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \exp\frac{2i\pi n x}{l}.$$

where

$$c_n = \frac{1}{l}\int_{-l/2}^{l/2} f(x)\exp\left(\frac{-2i\pi n x}{l}\right)dx$$

- The Fourier integral or Fourier transform:

$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty} dk\, \widetilde{f(k)} e^{ikx} \\ \widetilde{f(k)} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dx\, f(x) e^{-ikx} \end{aligned}$$