UNIVERSITY OF DUBLIN

2130

TRINITY COLLEGE

FACULTY OF SCIENCE

SCHOOL OF MATHEMATICS

SF Mathematics
SF Theoretical Physics
Foundation Scholarship

Hilary Term 2005

COURSE 231

Friday, April 1   Exam Hall   09.30 — 12.30

Dr. C. Ford

Attempt SIX questions

Log tables are available from the invigilators, if required.

Non-programmable calculators are permitted for this examination,—please indicate the make and model of your calculator on each answer book used.

1. Express $|\sin x|$ as a Fourier series.

2. Using the method of Frobenius, or otherwise, find the general solution of the ODE

$$xy'' + 2y'(x) - \alpha x y(x) = 0 \quad (\alpha \geq 0)$$

3. Determine the area and centroid of the part of the paraboloidal surface, $2z = x^2 + y^2$, below the $2z = 3$ plane.

4. Use a suitable change of variables to compute the integral

$$\int_{R^3} \mathrm{d}V \, \frac{e^{iaz} e^{-br}}{r}.$$

Here $r = \sqrt{x^2 + y^2 + z^2}$, a is a real constant and b is a positive constant.

A three dimensional version of the Fourier transform can be defined through

$$\tilde{\psi}(\mathbf{p}) = \frac{1}{(2\pi)^3} \int_{R^3} dV e^{-i\mathbf{p}\cdot\mathbf{r}} \psi(\mathbf{r})$$

where $\mathbf{p} = p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}$ and $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. This transform preserves spherical symmetry; if $\psi(\mathbf{r})$ is a function of $r = |\mathbf{r}|$ then $\psi(\mathbf{p})$ is a function of $p = |\mathbf{p}|$. Write e^{-br}/r as a three dimensional Fourier integral.

5. A function f is said to be an eigenfunction of the Fourier transform if it has the property

$$\tilde{f}(k) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \, e^{-ikx} f(x) = \lambda f(k)$$

where λ (the eigenvalue) is a constant.

Show that the Gaussian $f(x) = e^{-\frac{1}{2}x^2}$ is such an eigenfunction.

The Hermite polynomials H_n can be defined through the generating function

$$\Phi(x, h) = e^{2xh - h^2} = \sum_{n=0}^{\infty} \frac{h^n}{n!} H_n(x).$$

By computing the Fourier transform of $e^{-\frac{1}{2}x^2}\Phi(x,h)$, or otherwise, establish that the functions f_n defined by

$$f_n(x) = e^{-\frac{1}{2}x^2} H_n(x)$$

are eigenfunctions of the Fourier transform. Give the corresponding eigenvalues λ_n and comment on the result.

[The Gaussian integral

$$\int_{-\infty}^{\infty} dx\, e^{-ax^2+bx} = \sqrt{\frac{\pi}{a}} e^{b^2/4a}$$

$(a > 0, b \in \mathbb{C})$ may be quoted without proof]

6. Legendre's equation can be written as the eigenvalue problem

$$Ly \equiv -\frac{d}{dx}(1-x^2)\frac{d}{dx}y = \lambda y.$$

The operator L is Hermitian on the space of twice differentiable functions on $[-1,1]$; the eigenfunctions are real polynomials P_n $(n = 0, 1, 2, ...)$ with eigenvalues $\lambda_n = n(n+1)$. It is standard to normalise these Legendre polynomials so that

$$\int_{-1}^{1} dx P_m(x) P_n(x) = \frac{2\delta_{mn}}{2m+1}.$$

(i) Explain how to expand a function f defined on $[-1,1]$ as a Legendre series. Obtain a Parseval type formula for

$$\int_{-1}^{1} dx |f(x)|^2$$

(ii) The $\lambda = 0$ equation has the trivial solution $y(x) = P_0(x) = 1$. Verify that

$$y(x) = \log\frac{1+x}{1-x}$$

is a second solution. Explain why it is not an eigenfunction. Can it be expanded as a Legendre series?

7. In this question $\mathbf{F}$ is a smooth vector field defined in a connected domain $D \subset \mathbb{R}^3$.

(i) What is a conservative vector field? Prove that if $\mathbf{F}$ is conservative then

$$\oint_C \mathbf{F} \cdot \mathsf{d}\mathbf{l} = 0$$

for any closed piecewise smooth curve C in D.

(ii) Assume $\mathbf{F}$ has the property

$$\oint_S \mathbf{F} \cdot \mathsf{d}\mathbf{A} = 0$$

for every closed piecewise smooth surface S in D. Show that $\mathbf{F}$ is solenoidal. The converse of this statement is false. Establish this by demonstrating that the vector field

$$\mathbf{F} = \frac{\mathbf{r}}{r^3} \qquad \mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

is solenoidal in the domain $a < r < b \quad (b > a > 0)$ but $\oint_S \mathbf{F} \cdot \mathsf{d}\mathbf{A}$ does not vanish for all closed surfaces in the given domain.

8. Compute the line integral

$$\oint_C \frac{y dx - x dy}{x^2 + y^2}$$

where C is a circle of radius a (oriented anticlockwise) centred at the origin.

State Green's theorem in the plane (distinguish inner and outer boundary curves). Determine the line integral given above for an arbitrary closed piecewise smooth curve C in $\mathbb{R}^2$.