UNIVERSITY OF DUBLIN

2130

TRINITY COLLEGE

FACULTY OF SCIENCE

SCHOOL OF MATHEMATICS

SF Mathematics
SF Theoretical Physics
Foundation Scholarship

Hilary Term 2004

COURSE 231

Thursday, March 25 — Goldsmith Hall — 09.30 — 12.30

Dr. C. Ford

Attempt SIX Questions

1. Compute the area and centroid of the plane region enclosed by the cardioid $r(\theta) = 1 + \cos\theta$ (r and θ are polar coordinates).

2. Obtain the general solution of the ODE

$$y''(x) + 3y'(x) + 3y(x) = f(x)$$

where f is the periodic function defined by

$$f(x) = |x| - \frac{1}{2}\pi \quad \text{for} \quad -\pi < x < \pi \quad \text{and} \quad f(x + 2\pi) = f(x).$$

3. Determine the flux of the vector field $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ upwards through the part of the hyperboloid, $z^2 - x^2 - y^2 = 1$, lying between the $z = 1$ and $z = 2$ planes.

4. Prove that the Fourier transform of an even function is even. Is the Fourier transform of a real function always real? Determine the Fourier transform of $f(x) = e^{-a|x|} (a > 0)$. Hence or otherwise compute the integral

$$\int_{-\infty}^{\infty} dp \frac{\cos p}{1 + p^2}.$$

5. Define the curl of a vector field. Compute the curl of the vector field

$$\mathbf{F} = \frac{1}{2r(r-z)}(y\mathbf{i} - x\mathbf{j})$$

where $r = \sqrt{x^2 + y^2 + z^2}$.

Are $\mathbf{F}$ and curl $\mathbf{F}$ singular at the same points?

6. Obtain polynomial solutions of Hermite's equation

$$y''(x) - 2xy'(x) + 2\alpha y(x) = 0$$

for $\alpha = 1, 2$, and 3.

Consider the functions, $H_n(x), n = 1, 2, 3, \ldots$ defined through the generating function

$$\Phi(x,h) = e^{2xh-h^2} = \sum_{n=0}^{\infty} \frac{h^n}{n!} H_n(x).$$

Show that $H_n(x)$ satisfies Hermite's equation for $\alpha = n$. Prove that the $H_n(x)$ are polynomials.

7. Let $\mathbf{F}$ be a smooth vector field defined in some 3-dimensional domain D.

(i) Prove that if curl $\mathbf{F} = 0$ throughout D and D is simply connected then the line integral $\int_C \mathbf{F} \cdot \mathbf{dl}$ is path-independent, i.e., the same for any oriented piecewise smooth curve C joining two given points.

(ii) Show that the vector field

$$\mathbf{F} = \frac{y}{x^2+y^2}\mathbf{i} - \frac{x}{x^2+y^2}\mathbf{j}$$

has zero curl throughout the domain $x^2 + y^2 \geq \frac{1}{2}$. Establish that $\int_c \mathbf{F} \cdot \mathbf{dl}$ is not path-independent in this domain. Is there a scalar potential, ϕ, such that $\mathbf{F} = \text{grad } \phi$?

8. State Parseval's theorem for Fourier series. By repeated integration of the formula

$$x = -2\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin nx \qquad -\pi < x < \pi$$

or otherwise, obtain a Fourier series representation of x^3 for $-\pi < x < \pi$. Compute

$$\zeta(6) = \sum_{n=1}^{\infty} \frac{1}{n^6}$$