231 annual 2007 q9 outline solution

9 By separating variables, solve the two-dimensional Laplace equation for $\Phi(x,y)$ with what are called *Hadamard conditions*

$$\begin{aligned}\Phi(0,y) &= 0\\ \frac{\partial}{\partial x}\Phi(0,y) &= \frac{1}{n}\sin ny.\end{aligned}$$

*Solution:*This is a quick outline, I will skip over steps that are in the notes. So seperating the variables $\Phi(x,y) = X(x)Y(y)$ gives

$$X''Y + XY'' = 0 \tag{1}$$

or

$$\frac{X''}{X} = -\frac{Y''}{Y} \tag{2}$$

Since the left and right hand sides are functions of different independent variables, they must be equal to a constant, say E giving

$$\begin{aligned}X'' &= EX\\ Y'' &= -EY\end{aligned} \tag{3}$$

which each have solutions in the three classes as in the notes. Now we need to apply the boundary conditions,

$$\begin{aligned}\Phi(0,y) &= X(0)Y(y) = 0\\ \Phi(0,y) &= X'(0)Y(y) = \frac{1}{n}\sin ny\end{aligned} \tag{4}$$

so we need $X(0) = 0$ and $X'(0) =$constant. If this holds then we have

$$Y = \frac{A}{n}\sin ny \tag{5}$$

so $E = n^2$ and hence

$$X = B\sinh nx \tag{6}$$

where I have put substituted

$$X = C_1 e^{nx} + C_2 e^{-nx} \tag{7}$$

into the conditions on X and put the two exponentials together to get the $\sinh nx$. Now, putting this back together we get

$$\Phi(x,y) = \frac{C}{n}\sinh nx \sin nx \tag{8}$$